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Phase-winding solutions in a finite container above the convective threshold

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An analysis is presented of the steady states of two-dimensional convection near threshold in a laterally finite container with aspect ratio $2L \gg 1$. It is shown that the allowed wavevectors which can occur in the bulk of the container are reduced from a band $|q| \sim [(R-R_0)/R_0]^{\frac{1}{2}}$ in the laterally infinite system to a band $|q| \sim (R-R_0)/R_0$ in a system with sidewalls (R is the Rayleigh number and R_0 its critical value in the infinite system). The analysis involves an expansion of the hydrodynamic equations in the small parameter $[(R-R_0)/R_0]^{\frac{1}{2}}$, and leads to amplitude equations with boundary conditions, which generalize to higher order those previously obtained by Newell & Whitehead and Segel. The precise values of the allowed wavevectors depend on the Prandtl number of the fluid and the thermal properties of the sidewalls. For certain values of these parameters all the allowed wavevectors are less than the critical value q_0. The applicability of the results to convection in a rectangular container is briefly discussed.

1. Introduction

An interesting property of the steady states of convective flow in a laterally infinite Rayleigh–Bénard cell above threshold is the existence of a band of solutions with different lateral wavenumbers (Malkus & Veronis 1958; Schlüter, Lortz & Busse 1965; Joseph 1976). The experimental situation is not entirely clear, but there is evidence for a narrower band than predicted by the laterally infinite analysis, with a trend towards longer wavelengths as the Rayleigh number increases (for a review see Koschmieder 1974). Since real convecting systems have finite extent, it is important to understand the effect of lateral boundaries on the steady states of flow, and on the allowed wavenumbers in particular. (Of course, finite geometries will induce Fourier components of the flow at a wide range of wavenumbers. The question we are addressing is the wavenumber of the local periodicity (i.e. the inverse of the roll width) in regions of the cell, well away from the sidewalls, where a locally periodic roll structure is indeed evident.)

A complete solution of the convection problem in finite cells, involving the full analysis of three-dimensional flows, is extremely difficult. As a first step, the present paper considers the question of wavenumber selection for two-dimensional motion of rolls between two rigid sidewalls parallel to the rolls (with separation $2L \gg 1$), for Rayleigh numbers R slightly above threshold ($(R-R_0)/R_0 \ll 1$, where R_0 is the critical Rayleigh number of the infinite system). The central result obtained is that the presence of the sidewalls, no matter how distant, severely restricts the possible wavenumbers which can occur in the bulk of the system. Specifically, the band of available wavenumbers q about the critical wavenumber q_0 is reduced from a size $|q| \sim [(R-R_0)/R_0]^{\frac{1}{2}}$ in the infinite system to the range $|q| \sim (R-R_0)/R_0$ for a system with sidewalls. The impossibility of going from the finite system to the infinite one for $L \to \infty$ is a consequence of the non-local effect of the boundary conditions at the sidewalls. (The time necessary for the influence of distant sidewalls to be felt in the interior grows as $L \to \infty$, thus restoring continuity to the physical problem. The present work concerns itself only with *static* solutions, however, in which case the non-uniformity of the limit $L \to \infty$ is a real effect.)

Our work provides the solution to the mathematical question of the allowed wavenumbers of stationary solutions near threshold for the two-dimensional problem posed. The most important limitation to the direct applicability of our result to experimental situations is, of course, the neglect of the boundaries transverse to the convective rolls. These boundaries must in fact be more closely spaced than the longitudinal (short) boundaries whose effect we consider (Davis 1967; Luijkx & Platten 1981; Dubois & Bergé 1978). We shall argue below, however, that our analysis is likely to apply to the finite cell, provided that the rolls are indeed parallel to the short sidewalls. Such patterns have been observed experimentally in rectangular containers (Dubois & Bergé 1978).

An elegant reformulation of the results of Schlüter *et al.* (1965) for the infinite case, valid in the vicinity of the threshold, was presented by Segel (1969) and Newell & Whitehead (1969). These authors separated the stream function ψ for the motion into a rapidly oscillating part e^{iq_0x}, together with a slowly varying complex envelope function $A_0(X)$:

$$\psi \propto i\epsilon^{\frac{1}{2}}[e^{iq_0x}A_0(X)-e^{-iq_0x}A_0^*(X)] \sin \pi z. \tag{1.1}$$

The envelope function in their work varies on the slow scale

$$X = x\epsilon^{\frac{1}{2}}, \tag{1.2}$$

where the small parameter ϵ is here defined as

$$\epsilon = (R-R_0)/18\pi^2. \tag{1.3}$$

The nonlinear equation satisfied by A_0 is then

$$A_0''+A_0-|A_0|^2 A_0 = 0, \tag{1.4}$$

where the prime denotes d/dX. The solution in which A_0 is a constant represents motion with the critical wavevector q_0.

The results of Schlüter *et al.* for the infinite system are easily reproduced using these equations. In the infinite system there are a set of 'phase-winding' solutions

$$A_0(X) = (1-Q^2)^{\frac{1}{2}} e^{iQX}, \tag{1.5}$$

providing

$$-1 \leqslant Q \leqslant 1. \tag{1.6}$$

In view of (1.2) this represents a band of wavevectors in the physical variables, of size

$$-\epsilon^{\frac{1}{2}} \leqslant q \leqslant \epsilon^{\frac{1}{2}}. \tag{1.7}$$

The formalism in terms of an envelope function is particularly convenient for studying the influence of sidewalls on the phase-winding solutions. Since all fluid velocities are zero at rigid sidewalls the magnitude of the envelope function becomes small near the sidewalls, and then recovers away from the walls over the lengthscale $X = O(1)$. The boundary conditions appropriate to (1.4) were obtained by Segel (1969) and somewhat more systematically by Daniels (1977). These are

$$A_0(X) = 0 \quad (X = \pm\delta), \tag{1.8}$$

where†

$$\delta = \epsilon^{\frac{1}{2}} L \tag{1.9}$$

is the half-length of the box in the units appropriate to the X-variable. It may be readily shown that (1.4) and (1.8) do not allow any phase-winding solutions, which means that the boundary conditions have suppressed the whole band of wavevectors (1.7), to the order of the expansion (i.e. wavevectors $q = O(\epsilon^{\frac{1}{2}})$).

In order to determine the band (if any) of allowed wavevectors remaining in the presence of rigid sidewalls it is necessary to include corrections to the boundary condition (1.8) at the next order in $\epsilon^{\frac{1}{2}}$. It then turns out that the amplitude equation (1.4) must also be expanded to higher orders in $\epsilon^{\frac{1}{2}}$. In fact, to calculate the phase-winding solutions different lengthscales are introduced in different regions of the system, and amplitude equations are derived for the variation on each lengthscale. This approach, which will be described in detail below, is the direct extension of the multiple-scale method of Newell & Whitehead (1969) and Segel (1969) to the higher order required here (see also Daniels 1978). It is interesting to note, however, that our results may all be obtained from a more general envelope function $\Phi(x)$ defined by

$$\psi(x) = i(4/\pi)\,[e^{iq_0 x}\,\Phi(x) - \text{c.c.}]\sin \pi z + \text{higher harmonics}. \tag{1.10}$$

By expanding the hydrodynamic equations in the *independent* small parameters $|\Phi|$, ϵ and d/dx, we obtain the generalized amplitude equation

$$0 = \frac{d^2\Phi}{dx^2} + \epsilon\Phi - |\Phi|^2\Phi - i\left\{k_1\,\epsilon\frac{d\Phi}{dx} + k_2\frac{d^3\Phi}{dx^3} - (k_3+k_4)\,|\Phi|^2\frac{d\Phi}{dx} - (k_3+k_5)\,\Phi^2\frac{d\Phi^*}{dx}\right\} + k_6\,\Phi|\Phi|^4 + \ldots, \tag{1.11}$$

where ... represents higher-order terms which we will not need, and the k_i are real numbers calculated in appendix A. The boundary conditions are then, to the required order,

$$0 = \Phi - \alpha_\pm\frac{d\Phi}{dx} - \beta_\pm\frac{d\Phi^*}{dx} \quad (x = \pm L). \tag{1.12}$$

The Φ-term in (1.12) corresponds to (1.8), and the derivative terms represent the leading correction in $\epsilon^{\frac{1}{2}}$.

If we put

$$\Phi(x) = \epsilon^{\frac{1}{2}} A_0(X), \quad X = \epsilon^{\frac{1}{2}} x \tag{1.13}$$

and keep only the lowest-order term in (1.11) (i.e. of order $\epsilon^{\frac{3}{2}}$) we regain the amplitude equation given before, (1.4). More generally we expand the function Φ as

$$\Phi = \epsilon^{\frac{1}{2}} A_0(x) + \epsilon A_1(x) + \ldots, \tag{1.14}$$

and choose a suitable scaling for x according to the region. Inserting (1.14) into (1.11)

† The reader is warned that the parameter δ used here and in Cross *et al.* (1980) differs from the one defined by Daniels (1977, 1978), where $\delta \equiv \epsilon L^2$.

and equating like orders in $\epsilon^{\frac{1}{2}}$ then leads to precisely the same amplitude equations as we will derive directly from the hydrodynamic equations in §4.

Since the higher-order analysis is quite involved, we shall first discuss the simpler equation (1.4) with a number of phenomenological boundary conditions which illustrate the effect of sidewalls on the flow. It is also convenient to study first a semi-infinite system with one rigid sidewall (e.g. the region $x \geqslant -L$, with sidewall at $x = -L$). The single wall is sufficient to restrict the band of allowed wavevectors, in a way which we calculate in detail below. The addition of the second wall at $x = +L$ then has the effect of further restricting the wavevectors within this band to a discrete set quantized roughly in units of π/L.

The first phenomenological boundary effect we consider (model I) is (1.4) with the boundary condition

$$A_0(X) = \sqrt{2}\,\lambda\, e^{i\theta_0} \quad (X = \pm\delta), \tag{1.15}$$

where $\lambda \ll 1$. This condition directly restrains the amplitude to a small, but non-zero, value at the boundary. It has been studied previously by Daniels (1977, 1978), Brown & Stewartson (1978) and Hall & Walton (1977). For the semi-infinite system we find that the wavevectors $q = \epsilon^{\frac{1}{2}}Q$ are restricted to the band

$$-\lambda\epsilon^{\frac{1}{2}} \leqslant q \leqslant \lambda\epsilon^{\frac{1}{2}}. \tag{1.16}$$

This is similar to the band (1.7) of the infinite system but narrower by the factor $\lambda \ll 1$. The addition of the second wall gives quantized wavevectors with a number $N_{\mathrm{I}} \sim \lambda L\epsilon^{\frac{1}{2}}/\pi$ in the band (1.16).

The second phenomenological model we study (model II) is (1.4) with the boundary condition

$$A_0 - \lambda\alpha_{\pm} A_0' - \lambda\beta_{\pm} A_0^{*\prime} = 0 \quad (X = \pm\delta), \tag{1.17}$$

where $\alpha_{\pm}$ and $\beta_{\pm}$ are complex constants of order unity and $\lambda \ll 1$. Again this restricts the magnitude at the boundary, $|A_0| \sim \lambda \ll 1$, but is more closely related to the realistic boundary condition (1.12). The results for this case are similar to those of model I, but with an *asymmetric* band of allowed wavevectors

$$q_- \leqslant q \leqslant q_+, \tag{1.18}$$

with

$$q_{\pm} \propto \lambda\epsilon^{\frac{1}{2}}. \tag{1.19}$$

The constants of proportionality in (1.19) are of order unity and depend on $\alpha_{\pm}$ and $\beta_{\pm}$ (they can be either positive or negative). Again, for the finite system the number of solutions is of order $N_{\mathrm{II}} \sim \lambda L\epsilon^{\frac{1}{2}}/\pi$, with quantized wavevectors in the same band.

Finally we analyse the true hydrodynamic problem with realistic sidewall boundary conditions. The form of the solutions turns out to be analogous to that in model II, with $\lambda \equiv \epsilon^{\frac{1}{2}}$, but it is not identical, since the corrections to the amplitude equation of order $\epsilon^{\frac{1}{2}}$ given by (1.11) modify the details of the behaviour. The essential result, which may be calculated for the semi-infinite system, is that the band of wavevectors is restricted to a range given by (1.18), with

$$q_{\pm} \sim (-\eta \pm 1)\,\epsilon, \tag{1.20}$$

where η is $O(1)$ and depends on the Prandtl number σ and the thermal properties of the sidewall. In the finite system of length $2L$ there are a number N of order $\epsilon L/\pi$ allowed states in the band, with discrete wavevectors quantized in units of π/L. In all cases the band of wavevectors is reduced by a factor of the order of the reduction in magnitude of the envelope function at the boundary, reflecting the reduced fluid velocities in this region. The propagation of the influence of the boundaries into the bulk of the fluid will be seen explicitly in the solutions.

As noted above, the applicability of the two-dimensional solutions to the realistic case with boundaries transverse to the rolls, is not obvious *a priori*. The idea behind our suggestion that the short sidewalls are more important for determining the allowed wavenumbers in the bulk of the cell than the transverse sidewalls is the following: the analysis of the laterally infinite system imposes the constraint of a constant average roll spacing in the dynamics, and consequently leads to a wide band of allowed wavenumbers. The presence of the longitudinal (short) sidewall relaxes this constraint by allowing the creation or destruction of complete rolls close to the sidewall. On the other hand, the transverse sidewalls do not allow this process for the parallel roll pattern considered. Although the transverse sidewalls are important in locally suppressing the convective flows, and are also closer to the bulk of the cell, we argue that they are not in fact predominant in reducing the allowed band of wavenumbers in the bulk of the cell. It is conceivable, however, although we find no evidence for this, that the transverse sidewalls may perturb the details of the allowed band.

Our analysis will not apply to the more complicated convection patterns in which the rolls are not parallel to one pair of lateral sidewalls in a rectangular cell, but rather vary in direction throughout the cell. Such patterns are often observed in large-aspect-ratio Rayleigh–Bénard cells. Our method of analysis is in fact better suited to problems in which there is no rotational symmetry of the hydrodynamic equations in the absence of lateral boundaries, so that the flow is necessarily two-dimensional near threshold. The Taylor vortices in Couette flow provide an example of such a situation, although the end boundary effects in experimental systems are more difficult to treat analytically in this system. Rayleigh–Bénard convection in metals in the presence of a magnetic field parallel to the rolls provides another example where two-dimensional motion occurs, even if the transverse sidewalls are far apart (Fauve & Libchaber 1981; Busse & Clever 1982). We would expect our theory to apply, at least semiquantitatively, to this situation.

In §2 the basic equations are displayed. Section 3 describes the derivation of the lowest-order amplitude equation, and uses this to place general bounds on allowed wavevectors if the amplitude is reduced anywhere in the fluid. Explicit solutions are constructed for models I and II to show the restricted band of wavevectors resulting from the phenomenological boundary conditions. The true hydrodynamic problem with realistic sidewall boundary conditions is treated in §4. Section 5 concludes with a discussion of further work to be done and a comparison with other authors. A number of detailed calculations are contained in the appendices. A summary of the present work was published earlier (Cross *et al.* 1980).

2. Basic equations

Throughout this paper we shall be concerned with steady solutions of the two-dimensional Oberbeck–Boussinesq equations which govern the motion under gravity of a fluid with density ρ, thermal diffusivity κ, kinematic viscosity ν and coefficient of thermal expansion α. These OB equations are

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z} = 0, \tag{2.1a}$$

$$u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z} = -\frac{\partial p}{\partial x}+\sigma\nabla^2 u, \tag{2.1b}$$

$$u\frac{\partial w}{\partial x}+w\frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z}+\sigma\nabla^2 w+\sigma T, \tag{2.1c}$$

$$u\frac{\partial T}{\partial x}+w\frac{\partial T}{\partial z}-Rw = \nabla^2 T, \tag{2.1d}$$

where $\nabla^2 = \partial^2/\partial x^2+\partial^2/\partial z^2$, with the lengthscales x, z measured in units of the vertical cell height d, and the horizontal and vertical velocity components $u(x,z)$, $w(x,z)$ and reduced pressure $p(x,z)$ rendered dimensionless with respect to the quantities κ/d, and $\rho\kappa^2/d^2$. Temperatures are scaled with $\kappa\nu/\alpha g d^3$, T being the perturbation from the basic conducting temperature profile $-Rz$. The parameters of the problem are then the Rayleigh number

$$R = \alpha(T_\ell - T_u)\,gd^3/\nu\kappa, \tag{2.2}$$

the Prandtl number

$$\sigma = \nu/\kappa \tag{2.3}$$

and the length $2L$ of the (rectangular) container. The quantities T_u and T_ℓ are the constant temperatures maintained at the upper and lower boundaries at $z = 0$ and $z = 1$, where for analytical convenience we assume free boundary conditions

$$T = \frac{\partial u}{\partial z} = w = 0 \quad (z = 0, 1). \tag{2.4}$$

At the sidewalls, on the other hand, it is essential for our treatment that we use the realistic rigid boundary conditions on the velocities

$$u = w = 0 \quad (x = \pm L). \tag{2.5}$$

A number of different assumptions will be made for the thermal properties. One case is that of a sidewall with thickness t_w and thermal conductivity K_w, which is thermally clamped to the upper and lower plates. As shown in appendix B this implies the boundary condition on the $n = 1$ Fourier component $T^{(1)}$ of $T(z)$

$$\frac{\partial T^{(1)}}{\partial x} = \pm\mu T^{(1)} \quad (x = \pm L), \tag{2.6}$$

where

$$\mu = \pi^2 K_w t_w/K, \tag{2.7}$$

and K is the thermal conductivity of the liquid. Under these circumstances there is no heat flow out of the sidewalls in the absence of convection, so that the transition near $R = R_0$ remains a 'perfect bifurcation'. If, on the other hand, one assumes a finite heat flow out of the sidewalls whenever $T_\ell \neq T_u$, then the boundary condition can be taken as (Daniels 1977)

$$\frac{\partial T}{\partial x} = g_\pm(z) \quad (x = \pm L), \tag{2.8}$$

where $g_\pm(z)$ are specified functions.

3. Lowest-order amplitude equation and phenomenological models

In this section we consider the lowest-order amplitude equation as derived by Newell & Whitehead (1969) and Segel (1969), with phenomenological boundary conditions defining models I and II.

3.1. *The amplitude equation and general bounds*

In the infinite system ($L=\infty$), the convective threshold is at

$$R_0 = \tfrac{27}{4}\pi^4, \tag{3.1}$$

and the OB equations (2.1) can be consistently expanded in the small parameter

$$\epsilon = (R-R_0)/18\pi^2 \tag{3.2}$$

in terms of the slow scale (Segel 1969; Newell & Whitehead 1969)

$$X = x\epsilon^{\frac{1}{2}}. \tag{3.3}$$

Let $\psi(x,z)$ be the stream function, in terms of which the velocities are given by

$$u = \frac{\partial \psi}{\partial z}, \tag{3.4a}$$

$$w = -\frac{\partial \psi}{\partial x}, \tag{3.4b}$$

and let us expand ψ in $\epsilon^{\frac{1}{2}}$:

$$\psi = \epsilon^{\frac{1}{2}}\psi_0 + \epsilon\psi_1 + \epsilon^{\frac{3}{2}}\psi_2 + \dots. \tag{3.5}$$

At lowest order, the function ψ_0 can be expressed as

$$\psi_0(x,z) = \frac{i4}{\pi}[A_0(X)\,e^{i\pi x/\sqrt{2}} - A_0^*(X)\,e^{-i\pi x/\sqrt{2}}]\sin \pi z. \tag{3.6}$$

In appendix A it is shown that the slowly varying envelope function $A_0(X)$ satisfies the amplitude equation (Segel 1969; Newell & Whitehead 1969)

$$A_0'' + A_0 - |A_0|^2 A_0 = 0, \tag{3.7}$$

where the prime denotes differentiation with respect to X.

In discussing the effects of boundaries on this equation it will be convenient to employ the representation of A_0 in terms of amplitude and phase (Newell & Whitehead 1969)

$$A_0(X) = r(X)\,e^{i\theta(X)}, \tag{3.8}$$

for which (3.7) reads

$$r'' - r\theta'^2 + r - r^3 = 0, \tag{3.9}$$

$$\frac{d}{dX}(r^2\theta') = 0. \tag{3.10}$$

These equations may be integrated:

$$\tfrac{1}{2}r'^2 + \frac{Q^2}{2r^2} + \tfrac{1}{2}r^2 - \tfrac{1}{4}r^4 = E, \tag{3.11}$$

$$r^2\theta' = Q, \tag{3.12}$$

where Q and E are constants of integration. The analogy between these equations and those of a classical particle in a centrosymmetric potential $U(r) = \frac{1}{2}r^2 - \frac{1}{4}r^4$ has been pointed out by Newell & Whitehead (1969).

In terms of the above representation, the influence of the boundaries on the form of the solutions deep in the interior may be understood quite simply: the boundary conditions restrict the values of Q and E, which are 'constants of the motion' for (3.9) and (3.10). In fact, general bounds may be derived on the phase winding in the bulk if the magnitude of the envelope function $A_0(X)$ reaches a minimum r_m

somewhere in the cell (usually near the boundaries). Equation (3.11) implies

$$2E \geqslant \frac{Q^2}{r_m^2} + r_m^2 - \tfrac{1}{2}r_m^4. \tag{3.13}$$

Now let r_M be the maximum value reached by r, in a region where $r' = 0$ (the core). Then

$$\frac{Q^2}{r_M^2} + r_M^2 - \tfrac{1}{2}r_M^4 = 2E, \tag{3.14}$$

or, using (3.13),

$$Q^2 \leqslant r_M^2 r_m^2 [1 - \tfrac{1}{2}(r_M^2 + r_m^2)]. \tag{3.15}$$

Maximizing the right-hand side with respect to r_M, we find

$$Q \leqslant \sqrt{\tfrac{1}{2}}\, r_m(1 - \tfrac{1}{2}r_m^2). \tag{3.16}$$

In the core, where $r = r_M$, (3.12) may be integrated to give

$$\theta(X) = \tilde{Q}X + \tilde{C}, \tag{3.17}$$

where

$$\tilde{Q} = Q/r_M^2, \tag{3.18}$$

with

$$\frac{Q}{r_M^2} < \frac{r_m}{\sqrt{2}\, r_M^2}(1 - \tfrac{1}{2}r_m^2). \tag{3.19}$$

Equations (3.16)–(3.19) show that the amount of phase winding in the core region (where $r_M \approx 1$) far from any boundary, is limited by the minimum value taken on by the magnitude r *anywhere in the cell*. This effect, which persists in the other models considered in this paper, is the source of the restriction imposed by a boundary on available wavenumbers in the core. To show the effect explicitly we now calculate the phase winding solutions of the phenomenological models I and II.

3.2. *Model I*

We use the boundary conditions

$$A_0 = \sqrt{2}\,\lambda\, e^{\pm i\theta_0} \quad (X = \pm\delta), \tag{3.20}$$

with λ satisfying

$$\lambda \ll 1. \tag{3.21}$$

With this boundary condition the amplitude r is explicitly made small at the boundaries. The boundary condition is related to the 'finite-heat-current' boundary condition (2.8) considered by Daniels (1977). For that case, and assuming a constant heat current, we would have to take

$$\lambda = g_1 \epsilon^{-\frac{1}{2}}/18\sqrt{2}\,\pi^3, \tag{3.22}$$

where g_1 is the $n = 1$ Fourier component of the functions $g_+(z) = -g_-(z)$ in (2.8), which are assumed independent of ϵ. Thus $\lambda \propto \epsilon^{-\frac{1}{2}}$ and the assumption (3.21) would only be satisfied for ϵ not too small. We prefer here to use the conditions (3.20) and (3.21) as a convenient phenomenology.

For use in later sections we define the envelope function $A_0(X)$ on the boundaries to be

$$A_0 = r_\pm\, e^{i\theta_\pm} \quad (X = \pm\delta). \tag{3.23}$$

The model studied here has

$$r_+ = r_- = \sqrt{2}\,\lambda, \tag{3.24}$$

$$\theta_+ = -\theta_- = \theta_0. \tag{3.25}$$

It is simpler to consider first a semi-infinite region, which we take to be the region $-\delta \leqslant X \leqslant \infty$ with the appropriate boundary condition from (3.20) at $X = -\delta$. The argument given earlier shows that the wavevector in the bulk is limited by $|Q| \lesssim \lambda \ll 1$, and the solution for the magnitude r may be explicitly calculated using $Q \sim \lambda$ as a small parameter. The phase variation is then given in terms of Q, the wavevector in the core, by (3.12). The solutions in a large but finite box $\delta \gg 1$ may be constructed by matching the appropriate parts of the solutions in two semi-infinite regions $-\delta \leqslant X \leqslant \infty$ and $-\infty < X \leqslant \delta$ at $X = 0$. The spread of allowed wavevectors Q will be seen to be the same as in the semi-infinite case, but the wavevectors Q are now restricted by a condition on the phase difference across the system:

$$\int_{-\delta}^{\delta} dX\, \theta'(X) = Q \int_{-\delta}^{\delta} dX\, r^{-2}(X) = \theta_+ - \theta_- + 2n\pi. \tag{3.26}$$

3.2.1. *Phase winding in the semi-infinite region.* The semi-infinite region is taken as $x \geqslant -L$. It is convenient to use a coordinate defined as

$$\tilde{X} = (x+L)\,\epsilon^{\frac{1}{2}} = X+\delta, \tag{3.27}$$

where for the finite box
$$0 \leqslant \tilde{X} \leqslant 2L\epsilon^{\frac{1}{2}}. \tag{3.28}$$

The semi-infinite system is then given by the limit $L \to \infty$ in (3.28). We divide the system into a boundary region $\tilde{X} = O(\lambda)$ and a core region $\tilde{X} = O(1)$ or larger. For small λ the equation for the magnitude r may be solved explicitly in these two regions, and the constants of integration determined by the boundary conditions and by matching between the two regions. (Because the method of solution here is different from that in §4, the division into 'core' and 'boundary' regions is also different in the two cases.)

(*a*) *Solution in the core.* Here the term $Q^2/2r^2$ in (3.11) may be neglected and then an evaluation of this equation well away from the boundary gives

$$E = \tfrac{1}{4}. \tag{3.29}$$

Equation (3.11) then becomes
$$r'^2 = \tfrac{1}{2}(1-r^2)^2, \tag{3.30}$$

with the solution
$$r = \tanh\,[(\tilde{X}+\tilde{X}_0)/\surd 2], \tag{3.31}$$

where $\tilde{X}_0$ is a constant of integration to be determined by matching to the boundary. Equations (3.8), (3.12) and (3.31) imply that far from the boundary ($\tilde{X} \gg 1$) the core solution is

$$A_0(\tilde{X}) = \exp\,[i(Q\tilde{X}+C)], \tag{3.32}$$

with C a constant.

(*b*) *Solution in the boundary region.* Here the r^2 and r^4 terms in (3.11) are negligible, and the magnitude satisfies

$$r' = \pm\left(\frac{1}{2}-\frac{Q^2}{r^2}\right)^{\frac{1}{2}}. \tag{3.33}$$

Since the quantity inside the parentheses in (3.33) must be positive for $r = r_-$, we may define a parameter γ_- by

$$Q = \surd\tfrac{1}{2}\, r_-\sin\gamma_- = \lambda \sin\gamma_- \tag{3.34}$$

(cf. (3.24)) and integrate (3.33) to find

$$r^2 = 2\lambda^2 + 2\lambda\tilde{X}\cos\gamma_- + \tfrac{1}{2}\tilde{X}^2, \tag{3.35}$$

where the $\pm$ sign is absorbed into $\cos\gamma_-$, with $|\gamma_-| \leqslant \pi$ such that $-\frac{1}{2}\pi \leqslant \gamma \leqslant \frac{1}{2}\pi$ corresponds to the magnitude r *increasing* initially as $\tilde{X}$ increases (i.e. moving away from the boundary).

(c) *Matching between core and boundary layer.* The two solutions (3.31) and (3.35) may be matched at $\tilde{X} = b\lambda$, where $b \gg 1$, to find

$$\tilde{X}_0 = 2\lambda \cos \gamma_- . \tag{3.36}$$

Equations (3.24) and (3.34) give the range of allowed wavevectors in the core of the semi-infinite region, namely

$$-\lambda \leqslant Q \leqslant \lambda, \tag{3.37}$$

since any value of γ_- in (3.34) gives a solution, defined by (3.31), (3.35) and (3.12) for the phase.

3.2.2. *Phase winding in the finite system.* To consider the finite system $-\delta \leqslant X \leqslant \delta$ (or $0 \leqslant 2\tilde{X} \leqslant 2\delta$) we must evaluate the phase-winding integral (3.26) over the whole cell. For the half-cell $X < 0$, the magnitude $r(\tilde{X})$ remains as given by (3.31) and (3.35), since the corrections due to the boundary at $X = +\delta$ are negligible here. We then find contributions to the phase winding over the half-cell, given by the integral over the boundary region

$$\int_0^{b\lambda} d\tilde{X}\, \theta'(\tilde{X}) = \lambda \sin \gamma_- \int_0^{b\lambda} d\tilde{X} (2\lambda^2 + 2\lambda \tilde{X} \cos \gamma_- + \tfrac{1}{2}\tilde{X}^2)^{-1}$$
$$= \gamma_- - 2b^{-1} \sin \gamma_- + O(b^{-2}), \tag{3.38}$$

and the integral over the core

$$\int_{b\lambda}^{\delta} d\tilde{X}\, \theta'(\tilde{X}) = \lambda \sin \gamma_- \int_{b\lambda}^{\delta} d\tilde{X} \{\tanh [(\tilde{X} + \tilde{X}_0)/\sqrt{2}]\}^{-2}$$
$$= \lambda\delta \sin \gamma_- + 2b^{-1} \sin \gamma_- + O(b^{-2}), \tag{3.39}$$

with additional corrections of relative order λ. Thus, letting $b \to \infty$, the total phase winding over the half-cell is

$$\int_0^{\delta} d\tilde{X}\, \theta'(\tilde{X}) = \gamma_- + \lambda\delta \sin \gamma_- . \tag{3.40}$$

The same procedure may now be repeated for $X > 0$, and the resulting equations combined to yield

$$\theta_+ - \theta_- + 2n\pi = (\gamma_+ + \gamma_-) + 2\delta Q, \tag{3.41}$$

where γ_+ is the parameter analogous to γ_- but defined for $X > 0$, with $|\gamma_+| \leqslant \frac{1}{2}\pi$ once again corresponding to the amplitude increasing away from the boundary. The relations

$$Q = \lambda \sin \gamma_+ = \lambda \sin \gamma_- \tag{3.42}$$

then lead to two classes of solutions.

(i) $\gamma_+ = \gamma_- = \gamma$, so that (3.41) gives an implicit equation for Q:

$$Q = \pm \lambda \sin (Q\delta - \theta_0), \tag{3.43}$$

where + or − corresponds to n in (3.41) odd or even respectively.

(ii) $\gamma_+ + \gamma_- = \pm\pi$, so that (3.41) becomes

$$2\delta Q_n = 2\theta_0 + (2n+1)\pi, \tag{3.44}$$

with the range of Q limited by (3.42) to

$$|Q_n| < \lambda. \tag{3.45}$$

The evolution of the solutions $Q\delta$ of (3.43)–(3.45) may be calculated as a function of $\lambda\delta$. For $\lambda\delta$ small there are only two allowed wavevectors. As $\lambda\delta$ increases, more solutions appear. In fact there exists a band of solutions with

$$-\lambda \leqslant Q \leqslant \lambda, \tag{3.46}$$

as in the semi-infinite case, but with Q quantized to give roughly $6\lambda\delta/\pi$ values in the band. The precise numeration of the solutions for each wavevector is rather involved, and we will not discuss it for model I.

We should make some remarks on the accuracy of the solutions obtained. The phase-winding integral (3.41) defining the allowed values of Q has contributions $O(1)$ and $O(\delta Q)$. We calculate $r(X)$ to an accuracy sufficient to give these terms correctly to relative order λ, so that the values $Q = Q_n$ are themselves given to this relative accuracy (i.e. $Q = Q_n(1+O(\lambda))$). It should be noted that for very large δ, e.g. $\delta = O(\lambda^{-2})$, or for the semi-infinite system, the actual solution for the phase $\theta(X)$ will be in error by $O(1)$ far from the boundaries. If the detailed behaviour of $\theta(X)$ is required in this situation, a higher-order expansion must be performed. Also, in our solution we assumed that the magnitude saturates in the core, which requires $\delta \gg 1$. When δ decreases, e.g. $\delta < O(\lambda^{-1})$, only two of the phase-winding solutions (3.43)–(3.45) survive, and these presumably join up with the solutions studied by Daniels (1977, 1978) for $\delta = O(1)$.

3.3. *Model II*

We also wish to study the effect of the boundary condition

$$A_0 - \lambda\alpha_{\pm} A_0' - \lambda\beta_{\pm} A_0^{*\prime} = 0 \quad (X = \pm\delta), \tag{3.47}$$

where

$$\alpha_+ = -\alpha_-^* = \alpha, \quad \beta_+ = -\beta_-^* = \beta \tag{3.48}$$

are specified complex numbers of order unity and λ is a small real parameter. The motivation behind this boundary condition is twofold. Firstly, from a phenomenological point of view, (3.47) is the most general linear homogeneous boundary condition for an equation such as the amplitude equation. (The requirement of linearity is a consequence of the small velocities near a rigid boundary. The homogeneity maintains a perfect bifurcation at $\epsilon = 0$.) Secondly, as discussed earlier, the boundary conditions in the realistic case formally correspond to taking $\lambda = \epsilon^{\frac{1}{2}}$ in (3.47). The simpler calculation performed here illustrates the main effects found in §4.

The method of solution is similar to that used for model I except that now the boundary values of r are not given explicitly, but are functions of λ, α, β which must be determined. Again we introduce parameters $\gamma_{\pm}$ defined by

$$Q = \surd\tfrac{1}{2}\, r_+ \sin\gamma_+ = \surd\tfrac{1}{2}\, r_- \sin\gamma_-, \tag{3.49}$$

with $-\pi \leqslant \gamma_{\pm} \leqslant \pi$ chosen so that if r increases moving away from the boundary, then $|\gamma_{\pm}| \leqslant \frac{1}{2}\pi$.

If we first consider the semi-infinite case $-\delta \leqslant X \leqslant \infty$ with the $X = -\delta$ condition of (3.47), we have only one condition

$$\rho_- + \surd\tfrac{1}{2}\,\alpha^* \exp(i\gamma_-) + \surd\tfrac{1}{2}\,\beta^* \exp[-i(\gamma_- + 2\theta_-)] = 0, \tag{3.50}$$

where we write

$$r_{\pm} = \lambda\rho_{\pm}, \tag{3.51}$$

with $\rho_{\pm}$ real, and, since A_0' is typically $O(1)$, we expect $\rho_{\pm}$ to be $O(1)$. Equations (3.49)–(3.51) are simply solved by taking real and imaginary parts of (3.50) multiplied by exp $(-i\gamma_-)$. The imaginary part gives

$$2Q = -\lambda\alpha_i + \lambda|\beta| \sin(2\theta_- + \gamma_- + \phi_\beta), \tag{3.52}$$

which, since θ_- is arbitrary, allows a continuum of wavevectors in the band $Q_- \leqslant Q \leqslant Q_+$ with

$$Q_\pm = \tfrac{1}{2}\lambda[-\alpha_i \pm |\beta|], \tag{3.53}$$

where $\alpha_i = \operatorname{Im}\alpha$ and we have written $\beta = |\beta| \exp(i\phi_\beta)$. (It may be readily checked that the real part is also satisfied for any such Q.)

To consider the finite system $-\delta \leqslant X \leqslant \delta$ we must add the boundary condition at $X = \pm\delta$

$$\rho_+ + \sqrt{\tfrac{1}{2}}\,\alpha \exp(-i\gamma_+) + \sqrt{\tfrac{1}{2}}\,\beta \exp[i(\gamma_+ - 2\theta_+)] = 0, \tag{3.54}$$

together with the phase-winding integral as in (3.41)

$$(\theta_+ - \theta_-) + 2n\pi = (\gamma_+ + \gamma_-) + 2\delta Q. \tag{3.55}$$

The variables $\gamma_\pm$, $\rho_\pm$ and $\theta_\pm$ may be eliminated from the seven real equations given by (3.45)–(3.55), to give values for Q. The algebra is displayed in appendix C, where solutions are found with either:

$$\text{(i)}\quad 2Q = \lambda[-\alpha_i \pm |\beta| \sin(2Q\delta - \phi_\beta)]; \tag{3.56}$$

or

$$\text{(ii)}\quad 2Q_n\delta = \phi_\beta + (n+\tfrac{1}{2})\pi, \tag{3.57a}$$

with

$$|2Q_n + \lambda\alpha_i| \leqslant \lambda|\beta|. \tag{3.57b}$$

Again the wavevectors lie in the same band as in the semi-infinite case (3.53), but are further restricted to values quantized in units roughly of π/δ. The explicit values of Q resulting from (3.56) for given δ are easily found by writing

$$\lambda\delta = \frac{2Q\delta}{-\alpha_i \pm |\beta| \sin(2Q\delta - \phi_\beta)}. \tag{3.58}$$

The allowed band of wavevectors depends crucially on the parameter

$$\eta \equiv \alpha_i/|\beta|. \tag{3.59}$$

For $\eta < -1$, (3.58) implies that $Q > 0$, whereas, for $\eta > 1$, $Q < 0$, and, when $-1 < \eta < 1$, Q can take on both positive and negative values. We defer discussion of the numerical evaluation of (3.58) to §4.2.2.

4. Realistic boundary conditions at the sidewalls

Let us now consider the full hydrodynamic problem (2.1) with the physical boundary conditions (2.4)–(2.6). Again it is easiest to solve first the semi-infinite problem $x \geqslant -L$.

4.1. *Regions and amplitude equations*

We define three different regions, in which solutions may be explicitly calculated. (As noted earlier, these regions are chosen in a different manner than for the models.)

(*i*) *Sidewall region:*

$$\tilde{x} = L + x = O(1). \tag{4.1}$$

In this region the amplitude is small and the hydrodynamic equations may be linearized.

(*ii*) *Boundary layer:*

$$\tilde{X} = (L+x)\,\epsilon^{\frac{1}{2}} = O(1). \tag{4.2}$$

Here amplitude equations for an envelope function with a spatially varying magnitude may be obtained and solved. The matching between regions (i) and (ii) gives boundary conditions on the envelope function.

(*iii*) *Core region:*

$$\bar{X} = (L+x)\,\epsilon = \tilde{X}\epsilon^{\frac{1}{2}} = O(1). \tag{4.3}$$

In the region $\tilde{X} \gg 1$, $\bar{X} = O(1)$, the magnitude of the envelope function has saturated, but the phase variation is significant. As explained below, the scaling of $\bar{X}$ given in (4.3) follows from the matching of the solution in region (iii) to that in region (ii).

For the realistic system it is not possible to define constants of the motion that relate the solutions in the core directly to those near the boundary where the velocities are reduced. Instead, the propagation of the influence of the boundaries on the wavevectors in the core is given by successive matchings between regions. This difference in approaches accounts for the different definitions of the regions here and in §3. In the sidewall region (i) the stream function ψ is small and the hydrodynamic equations (2.1) may be linearized. The solutions including the physical boundary conditions are shown in (B 9)–(B 11) (appendix B). In the boundary layer (region (ii)) the starting equations (2.1) may be expanded in $\epsilon^{\frac{1}{2}}$, treating $d/d\tilde{X}$ as $O(1)$. This procedure is carried out explicitly to $O(\epsilon^2)$ in appendix A, and one obtains the amplitude equations (Daniels 1978)

$$A_0'' + A_0 - A_0\,|A_0|^2 = 0, \tag{4.4}$$

$$A_1'' + A_1 - 2A_1\,|A_0|^2 - A_1^*\,A_0^2 + F_{\frac{1}{2}}[A_0] = 0, \tag{4.5}$$

where
$$F_{\frac{1}{2}}[A_0] = -i\{k_1\,A_0' + k_2\,A_0''' - (k_3+k_4)\,|A_0|^2 A_0' - (k_3+k_5)\,A_0^2\,A_0^{*\prime}\}, \tag{4.6}$$

and the numbers k_i are defined in (A 21). The function A_1 is defined analogously to A_0, but from the function ψ_1 of (3.5) (see (A 12)).

Boundary conditions on the amplitude equations (4.4)–(4.6) for $\tilde{X} \rightarrow 0$ are obtained by matching to the asymptotic solution in the sidewall region as $\tilde{x} \rightarrow \infty$ (see appendix B). The result obtained there reads (Daniels 1978)

$$\left.\begin{aligned} A_0 &= 0 \quad &(4.7a)\\ A_1 - \alpha_-\,A_0' - \beta_-\,A_0^{*\prime} &= 0 \quad &(4.7b)\end{aligned}\right\}\quad (\tilde{X} = 0),$$

where α_- and β_- are $O(1)$ complex numbers whose values are listed in (B 15) and (B 11). The behaviour in the core is found by expanding the starting equations in $\epsilon^{\frac{1}{2}}$, treating $d/d\bar{X}$ as $O(1)$. This is done in appendix E.

4.2. *Phase-winding solutions*

4.2.1. *Semi-infinite system.* The behaviour in the boundary region (ii) is given by (4.4)–(4.6) with the boundary conditions (4.7*a*, *b*) at $\tilde{X} = 0$. The solution of (4.4) is

$$A_0 = e^{i\phi_-}\tanh\,(\tilde{X}/\surd 2), \tag{4.8}$$

where ϕ_- is an arbitrary real constant. When this expression for A_0 is inserted into (4.5) a second-order, *linear* inhomogeneous equation is obtained for A_1. The solution is found in appendix D, and reads

$$A_1 = e^{i\phi_-}[a_-\,\mathrm{sech}^2\,(\tilde{X}/\surd 2) + i\{c_-\,\tanh\,(\tilde{X}/\surd 2) + d_-[\surd\tfrac{1}{2}\,\tilde{X}\tanh\,(\tilde{X}/\surd 2) - 1] + \surd\tfrac{1}{2}\,B(\tilde{X})\}], \tag{4.9}$$

where a_-, c_- and d_- are real $O(1)$ constants to be determined, and the particular integral $B(\tilde{X})$ is written down explicitly in (D 12). For the present calculation we need only know that

$$B(0) = b, \tag{4.10a}$$

where b is given in (D 13), and that

$$B(\tilde{X}) < \infty \quad \text{for} \quad \tilde{X} \to \infty, \tag{4.10b}$$

so that A_1 is dominated by the third term on the right-hand side of (4.9) in this limit. The boundary condition (4.7b) at $\tilde{X} = 0$ implies

$$(a_- - id_- + ib/\sqrt{2} + \alpha^*/\sqrt{2})\, e^{i\phi_-} + (\beta^*/\sqrt{2})\, e^{-i\phi_-} = 0. \tag{4.11}$$

Let us now consider the envelope function in the limit $\tilde{X} \to \infty$, where (4.8) and (4.9) yield

$$A_0 + \epsilon^{\frac{1}{2}} A_1 \underset{\tilde{X}\to\infty}{\sim} e^{i\phi_-}[1 + i(d_-/\sqrt{2})\, \epsilon^{\frac{1}{2}} \tilde{X}]. \tag{4.12}$$

Equation (4.12) implies that for $\tilde{X} \gg \epsilon^{-\frac{1}{2}}$ the first-order 'correction' dominates the zeroth-order term and the boundary-layer expansion breaks down. This suggests the existence of a new regime (the core) where variations on the scale $\bar{X} = \epsilon^{\frac{1}{2}}\tilde{X}$ are of order unity (cf. (4.3)). In appendix E it is shown that with this scaling the envelope function has the simple form

$$A_0(\bar{X}) = \exp i[\bar{Q}\bar{X} + \bar{C}], \tag{4.13}$$

i.e. it has a constant amplitude and a phase which is linear in $\bar{X}$.

In order to find the allowed values of the wavevector $\bar{Q}$ we shall match the core solution (4.13) for $\bar{X} \to 0$ to the asymptotic form of the boundary-layer solution (4.8), (4.9) as $\tilde{X} \to \infty$. Expanding (4.13) for small $\bar{X}$ gives

$$A_0(\bar{X}) \simeq e^{i\bar{C}}[1 + i\bar{Q}\bar{X}] = e^{i\bar{C}}[1 + i\bar{Q}\epsilon^{\frac{1}{2}}\tilde{X}], \tag{4.14}$$

where we have used (4.3). In the limit $\tilde{X} \to \infty$ the first term matches to A_0, (4.8), to give

$$\phi_- = \bar{C}, \tag{4.15}$$

and the second term, which is of relative order $\epsilon^{\frac{1}{2}}$, matches to A_1, (4.9), to give

$$\bar{Q} = \sqrt{\tfrac{1}{2}}\, d_-. \tag{4.16}$$

Defining the wavevector Q in the core by

$$Q\tilde{X} = \bar{Q}\bar{X} = \bar{Q}\tilde{X}\epsilon^{\frac{1}{2}}, \tag{4.17}$$

we obtain

$$Q = \epsilon^{\frac{1}{2}} d_-/\sqrt{2}. \tag{4.18}$$

The fact that we have been able to match the boundary-layer variation (4.8), (4.9) with the core behaviour (4.13) confirms the correctness of our choice for the core scaling (4.3).

The imaginary part of (4.11) gives

$$2Q = \epsilon^{\frac{1}{2}}[-\alpha_i + b - |\beta| \sin(2\bar{C} + \phi_\beta)], \tag{4.19}$$

and, since $\bar{C}$ is arbitrary, we obtain a continuum of solutions bounded by limiting curves $Q_\pm(\epsilon)$. The solution is thus quite analogous to that of model II (3.52) except for the replacements

$$\alpha_i \to \tilde{\alpha}_i = \alpha_i - b, \tag{4.20}$$

$$\lambda \to \epsilon^{\frac{1}{2}}. \tag{4.21}$$

In the original variable x the band of available wavevectors q is given by

$$q_\pm = -\tfrac{1}{2}|\beta|\,(\eta \mp 1)\,\epsilon, \tag{4.22}$$

with
$$\eta = (\alpha_i - b)/|\beta|. \tag{4.23}$$

Inserting the values of α, β and b from (B 15) and (D 14), we obtain (Cross *et al.* 1980)

$$\eta = (32\sqrt{3})^{-1}(5+21\sigma^{-1}+40\sigma^{-2})(1+4\bar{\mu}+6\bar{\mu}^2)^{-\frac{1}{2}}. \tag{4.24}$$

It follows from (4.22) and (4.23) that $q_- < 0$, but q_+ can be positive or negative depending on the magnitude of σ and $\bar{\mu} = (1+2\mu/\pi)^{-1}$.

4.2.2. *Finite system.* For the cell of length L the solutions may be obtained by repeating the analysis of §4.2.1 starting from the boundary at $x = L$ (or $X = \delta$). The equation corresponding to (4.11) is found to be

$$(a_+ - id_+ - ib/\sqrt{2} + \alpha/\sqrt{2})\, e^{i\phi_+} + (\beta/\sqrt{2})\, e^{-i\phi_+} = 0, \tag{4.25}$$

and the core solution may still be taken in the form (4.13), with

$$\phi_+ - 2Q\delta = \bar{C}, \tag{4.26}$$

$$Q = -\epsilon^{\frac{1}{2}} d_+/\sqrt{2}. \tag{4.27}$$

Equations (4.11), (4.15), (4.18), (4.25)–(4.27) consist of 8 equations for the 8 real constants $a_\pm$, $d_\pm$, $\phi_\pm$, $\bar{C}$ and Q, with b, α, β, δ and ϵ known parameters. By a calculation similar to that in appendix C the above system may be reduced to

$$|\beta| \sin(\phi_- + \phi_+)\cos(2Q\delta - \phi_\beta) = 0, \tag{4.28a}$$

$$2Q = \epsilon^{\frac{1}{2}}[-\alpha_i + b + |\beta|\cos(\phi_- + \phi_+)\sin(2Q\delta - \phi_\beta)]. \tag{4.28b}$$

The solutions are as in (3.56) and (3.57):

$$\text{(i)}\quad 2Q = \epsilon^{\frac{1}{2}}[-\alpha_i + b \pm |\beta| \sin(2Q\delta - \phi_\beta)], \tag{4.29}$$

$$\text{(ii)}\quad 2Q_n\delta = \phi_\beta + (n+\tfrac{1}{2})\pi, \tag{4.30a}$$

with
$$|2Q_n + \epsilon^{\frac{1}{2}}(\alpha_i - b)| < \epsilon^{\frac{1}{2}}|\beta|. \tag{4.30b}$$

Equations (4.29) and (4.30) may be rewritten in terms of the physical variables as

$$\text{(i)}\quad 2qL = \epsilon L|\beta|\,[-\eta \pm \sin(2qL - \phi_\beta)], \tag{4.31}$$

$$\text{(ii)}\quad 2q_n L = \phi_\beta + (n+\tfrac{1}{2})\pi, \tag{4.32a}$$

$$|2qL + |\beta|\,\epsilon L\eta| < |\beta|\,\epsilon L, \tag{4.32b}$$

with η the parameter defined in (4.23). The solutions of (4.31) and (4.32), shown in figure 1, are most simply obtained by plotting ϵL as a function of qL for given values of η and β. For small ϵL there are only two values of qL, corresponding to the solutions that evolve from the linear onset at $\epsilon = \frac{1}{4}\pi^2 L^{-2}$, as studied by Drazin (1975) and Daniels (1977). (For each value of q there are two solutions related by $\psi \to -\psi$.) As ϵL increases more wavevectors successively appear. Note that the amplitude of each solution is finite at the bifurcation: in fact it has an envelope function with magnitude close to saturation in the core. The evolution of a particular solution as ϵ increases or decreases is quite different depending on whether η (4.23) is greater or less than unity. For $\eta < 1$ (case (*a*) of figure 1), the stationary solution to the hydrodynamic equations may evolve continuously as ϵ is increased. In contrast, for $\eta > 1$ (case (*b*) of figure 1) there are necessarily discontinuities in this evolution as ϵ increases (decreases), corresponding physically to the sudden disappearance (appearance) of rolls. Of course to determine the actual evolution of the convection pattern in a physical experiment when the parameters pass outside the stationary band requires

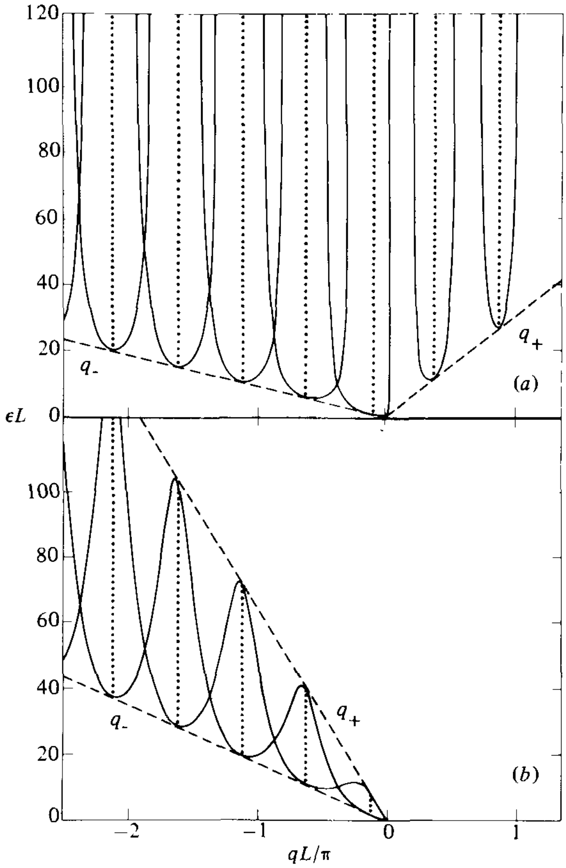

FIGURE 1. Allowed wavevectors for steady two-dimensional motion in a rectangular container with aspect ratio $2L \gg 1$, near threshold. The reduced Rayleigh number is plotted *vs.* the deviation of the wavevector from its critical value q_0, as given by (4.31) and (4.32). The solid lines correspond to (4.31), and the dotted lines to (4.32). All solutions are confined to the interval $q_- < q < q_+$, with $q_\pm$ given by the dashed lines. In case (*a*) $(q_+ > 0)$, the parameter η defined in (4.23) satisfies $\eta < -1$, and the wavevector can vary continuously with increasing ϵ. In case (*b*) $(q_+ < 0)$, we have $-1 < \eta < 1$, and q necessarily changes discontinuously as ϵ is increased or decreased. The parameters η and β in (4.31) and (4.32) for these plots correspond to $\sigma = 0{\cdot}78$, and to perfectly insulating sidewalls $(\mu = 0)$ for case (*a*), and perfectly conducting sidewalls $(\mu = \infty)$ for case (*b*). The present figure also yields solutions for model II given in (3.56)–(3.59) for the same values of η and β, provided that the abscissa and ordinate are changed to $Q\delta/\pi$ and $\lambda\delta$ respectively.

an analysis of the full time-dependent equations, which goes beyond the scope of this work.

5. Discussion and conclusion

We have shown that the presence of rigid sidewalls severely restricts the band of allowed wavevectors $q_- \leqslant q \leqslant q_+$ for two-dimensional flow near onset from that suggested by the analysis in an infinite system. In fact the band is reduced from a width of order $[(R/R_0)-1]^{\frac{1}{2}}$ to one of order $(R/R_0)-1$. Furthermore, for the system we consider, we find $q_- < 0$ always, but q_+ may be either greater or less than zero, depending on the physical parameters. In the former case a particular solution may evolve continuously as the Rayleigh number is increased. In the latter case the wavevector in the core must decrease as the Rayleigh number increases, and this occurs by discontinuous jumps corresponding to the loss of a roll at the sidewalls.

The reduction in the band of allowed wavevectors arises from considering the possible *stationary* solutions: we do not, in this paper, study the more difficult question of *stability* of these solutions. One of us (Daniels 1981) has analysed the stability of the solutions to two-dimensional disturbances (no variation transverse to the rolls). The result may be easily summarized: out of the stationary solutions defined by (4.29) and (4.30) those solutions out of class (i) with negative derivative with respect to Q of the right-hand side of (4.29) (i.e. $d(Q/\epsilon^{\frac{1}{2}})/dQ < 0$) are stable. All other solutions are unstable. The possibility of instabilities associated with transverse disturbances (three-dimensional motion) is an interesting question. This has been discussed in the infinite system, where the most important effect near threshold is the zigzag instability which for free–free horizontal boundary conditions eliminates all states with $q < -c_0\epsilon^2$, $c_0 = O(1)$ (Busse 1978; Joseph 1976). It is thus interesting that for small σ we find $q_+/\epsilon < 0$, i.e. *all* the states of steady two-dimensional flow of the finite system are in the band of unstable wavevectors in the infinite case. Furthermore, the band of wavevectors found is independent of the size of the system. It seems reasonable to expect that, in a large-enough system, the zigzag instability is not much suppressed, so that all two-dimensional states are then eliminated when $q_+ < 0$.

All our calculations have been for the physically unrealistic free–free horizontal boundary conditions. The rigid case is analytically much more difficult, though the lowest-order amplitude equation has been derived in that case too (Cross 1980), and presumably this can be continued to higher order, so that we expect similar effects to occur there. The values of $q_\pm$ as a function of σ and μ are, of course, expected to be different. The experimentally observed increase in wavelength with increasing Rayleigh number (Koschmieder 1974) would be accounted for by a q_+ less than zero for most values of σ. For the rigid–rigid case the threshold of the zigzag instability in the infinite system is $q < -c\epsilon$ with c positive and $O(1)$ (Busse 1978), so that in this case some of the stationary states we have calculated could be stable to fluctuations in the transverse direction.

As noted in §1, the most important limitation to the applicability of our work is our neglect of the boundaries transverse to the rolls. These boundaries must in fact be more closely spaced than the longitudinal (short) boundaries whose effect we consider (Davis 1967; Luijkx & Platten 1981; Dubois & Bergé 1978). Since a solution of the full three-dimensional problem in a finite cell is not available at present, we must use heuristic arguments to assess the effect of the transverse walls on our two-dimensional solutions.

Let us consider a rectangular container of width $2M$ in the y-direction, and introduce the coordinate

$$Y = (2q_0)^{\frac{1}{2}}\, y\epsilon^{\frac{1}{4}}. \tag{5.1}$$

Then the lowest-order amplitude equation is (Newell & Whitehead 1969)

$$(\partial_X - i\partial_Y^2)^2 A_0 + A_0 - |A_0|^2 A_0 = 0, \tag{5.2}$$

with boundary conditions (4.7*a*) at $x = \pm L$, and (Brown & Stewartson 1977)

$$\partial_Y A_0 = A_0 = 0 \quad \text{for} \quad Y = \pm(2q_0)^{\frac{1}{2}} M\epsilon^{\frac{1}{4}}. \tag{5.3}$$

These equations incorporate the full three-dimensionality of the flow at this order in ϵ. Appealing to the experimental fact (Dubois & Bergé 1978) that flows with straight rolls in the y-direction occur when $M < L$, we shall look for solutions of (5.2) and (5.3) whose phase is independent of Y. In the core region (4.3) the magnitude $|A_0|$ is independent of X, and its phase is linear in X (see (4.13)), so (5.2) leads to a phase independent of Y and an equation for the magnitude

$$-(\partial_Y^2-Q)^2|A_0|+|A_0|-|A_0|^3=0, \tag{5.4}$$

with the boundary conditions (5.3). This equation is a generalization to finite Q of equation (6.53) of Brown & Stewartson (1977). It is clear that a solution exists with $|A_0|\approx 1$ in the bulk of the cell, which falls off to zero in a length $\delta Y=O(1)$ ($\delta y=O(\epsilon^{-\frac{1}{4}})$) near the boundaries at $y=\pm M$. Thus the phase-winding solutions found in §4 appear to be compatible with the transverse boundary conditions over the whole core region, at least in lowest order. A more difficult problem arises in the boundary layer (4.2) where $|A_0|$ depends on X, and (5.4) must be modified even in lowest order. Our conjecture is that there still exists a region far from the boundaries at $y=\pm M$, where the Y-dependence of A_0 can be neglected, and where the matching carried out in §4 remains valid. Though we have no concrete argument to justify this conjecture, we see no particular reason why the transverse boundary conditions should affect A_0 arbitrarily far from $y=\pm M$. The boundary conditions at $x=\pm L$, on the other hand, act on the phase of A_0, and their influence extends throughout the cell. Clearly, the above arguments and conjectures must be considered tentative, until a more systematic analysis of the three-dimensional problem is achieved.

Experiments are often performed in cylindrical containers (Koschmieder 1974; Ahlers & Behringer 1978). If cylindrically symmetric patterns occur (as is sometimes observed), then there are bending effects which were not considered here, which appear to restrict the wavenumber more drastically than for straight rolls (Pomeau & Manneville 1981).

The influence of sidewalls on convective states near threshold has been previously studied. Our work has sought phase-winding solutions, which we find in the range $\epsilon=O(L^{-1})$. Indeed, the analysis fails when $\epsilon=O(L^{-2})$ since the width of the boundary-layer region at the walls, $O(\epsilon^{-\frac{1}{2}})$, is then comparable to the width of the container. Solutions in the range $\epsilon=O(L^{-2})$, which relate the initial development of the motion at the critical Rayleigh number to the solutions described in the present work, have been studied by Daniels (1978), who finds four solutions with thresholds at $\epsilon=\frac{1}{4}\pi^2L^{-2}+O(L^{-3})$. The solutions in the range $\epsilon=O(L^{-2})$ may again be represented by envelope functions A_0 and A_1, with A_0 written in terms of Jacobian elliptic functions. The magnitude of A_0 varies with X, but its phase remains constant. For comparison with Daniels (1977, 1978) it should be noted that the envelope-function expansion is done there using L^{-1} as the expansion parameter, rather than $\epsilon^{\frac{1}{2}}$ as used in the present work. Moreover, as noted earlier, Daniels defines a quantity $\delta=\epsilon L^2$, whereas the present paper has $\delta=\epsilon^{\frac{1}{2}}L$. For the case $\epsilon=O(L^{-2})$ considered there, the expansions are directly comparable with our work, the differences occurring only in normalization factors, which are $O(1)$. The form of A_1 is not needed in that work, but we can see from (4.8) and (4.9) that, as $\tilde{X}\to\infty$, A_1 becomes linear in $\tilde{X}$ with $O(1)$ coefficient, while A_0 approaches a constant $O(1)$ value. The solution fails when the 'correction' $\epsilon^{\frac{1}{2}}A_1$ becomes comparable to A_0, which occurs for $\tilde{X}\sim\epsilon^{\frac{1}{2}}L$. Since Daniels' treatment was for $\epsilon=O(L^{-2})$ this regime was never attained and A_1 remained small compared with A_0 away from the boundary. For the present case, however, it is precisely in the region $\tilde{X}\gtrsim\epsilon^{\frac{1}{2}}L$ that we enter a new regime, the core region, with a variable-phase solution for A_0. It may be readily verified that as $\epsilon L\to 0$ the solutions we have obtained correctly tend to the four linear onset solutions discussed by Daniels. Finally, we note that the phase-winding solutions we find for model I in the range $\epsilon = O(L^{-2})$, where the simple amplitude equation is adequate, exist only for $\lambda = O(1)$, a range of parameters not considered in the earlier investigation of Daniels (1977).

Daniels (1977) also considers solutions in which the amplitude goes to zero somewhere in the interior, but the phase is constant. These are the solutions that bifurcate linearly from the conducting solution for $\epsilon L^2 = \frac{1}{4}m^2\pi^2$, $m > 1$. Similar solutions in the infinite system, which are in addition to the phase-winding solutions discussed here, were considered by Segel (1969) and Newell & Whitehead (1969). In both cases these solutions were found to be unstable with respect to the phase. A more general set of solutions may be imagined in which fictitious walls are introduced in the interior, where the amplitude A_0 does not vanish, but merely drops to a small value. There is, however, no evidence that these solutions are of physical interest, in contrast with the phase-winding solutions, since the former seem likely to be unstable. The present work has therefore focused on the effect of sidewalls on solutions that have the simple phase-winding form in the core.

Pomeau & Manneville (1980) have studied the question of wavenumber selection in finite layers on the basis of two simplified models, involving fourth-order equations:

$$\text{model } (a): \quad \partial_t u = [\epsilon - (\partial_x^2 + q_0^2)^2]\, u - u^3, \tag{5.5a}$$

$$\text{model } (b): \quad \partial_t u = [\epsilon - (\partial_x^2 + q_0^2)^2]\, u - u\, \partial_x u, \tag{5.5b}$$

where $u(x, t)$ is a real function satisfying the boundary conditions

$$u(x,t) = \partial_x u(x,t) = 0 \quad (x = \pm L). \tag{5.6}$$

Pomeau & Manneville studied these models by numerical integration and found the following results: for both models (a) and (b) they claimed that a *single* wavenumber was always attained for given ϵ. For the case of model (a) the selected wavenumber followed from a variational principle, but no such principle exists for model (b). In a later publication Pomeau & Zaleski (1980) have attempted to determine the stationary solutions of (5.5) analytically by relating them to solutions of the linear eigenvalue equation

$$(\partial_x^2 + q_0^2)^2\, u^{(\lambda)} + \lambda u^{(\lambda)} = 0 \tag{5.7}$$

at $\epsilon = 0$. These authors also concluded that for each ϵ there exists a unique solution (apart from symmetries with respect to $x = 0$) without nodes in the envelope in the interior of the container.

It turns out that the models of Pomeau & Manneville can be treated with the methods of the present paper by considering them to be simplified versions of the 'microscopic' Boussinesq equations (2.1) with boundary conditions. Indeed, as mentioned in our earlier note (Cross *et al.* 1980), the calculations of appendices A and B may be repeated for (5.1) and amplitude equations of the form (4.4)–(4.6) derived. The calculation is performed in appendix F, where it is shown that phase-winding solutions exist for both of the models considered, with limiting wavevectors equal to

$$\text{model } (a): \quad q_{\pm} = \pm\epsilon/16q_0^3, \tag{5.8}$$

$$\text{model } (b): \quad q_{+} = -\epsilon 47/48q_0^3, \tag{5.9}$$

$$q_{-} = -\epsilon 53/48q_0^3. \tag{5.10}$$

In a cell of length $2L$ the number of solutions is of order

$$N \simeq \epsilon L,$$

just as for the more-realistic Boussinesq case. Model (*a*) corresponds to the case $q_+ > 0$ in figure 1, whereas model (*b*) has $q_+ < 0$. Thus the numerical results of Pomeau & Manneville (1980) are seen to be misleading, since more than one stationary solution of the equation exists for a given ϵ, in *both* models (*a*) and (*b*). The arguments of Pomeau & Zaleski (1980) are not sufficiently general to have found these other solutions. (*Note added in proof*: in a later publication Pomeau & Zaleski (1981) have reconsidered this problem and now agree with the result stated in Cross *et al.* (1980) and in (5.8)–(5.10).)

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Appendix A. Derivation of the amplitude equation

In this appendix we sketch the full derivation of the amplitude equations needed in our work. We use the methods of Newell & Whitehead (1969) and Daniels (1977, 1978). The principles of the derivation are well illustrated in the simpler examples treated in appendix F. From the equations of motion (2.1) and the definition (3.4) we obtain the basic equation for the stream function ψ:

$$\nabla^6\psi - R\frac{\partial^2\psi}{\partial x^2} = \sigma^{-1}\nabla^2\left[\frac{\partial}{\partial z}(uu_x + wu_z) - \frac{\partial}{\partial x}(uw_x + ww_z)\right] + \frac{\partial}{\partial x}(uT_x + wT_z). \quad \text{(A 1)}$$

Let us introduce the expansions

$$\psi = \epsilon^{\frac{1}{2}}\psi_0 + \epsilon\psi_1 + \epsilon^{\frac{3}{2}}\psi_2 + \dots, \quad \text{(A 2)}$$

$$u = \epsilon^{\frac{1}{2}}u_0 + \epsilon u_1 + \epsilon^{\frac{3}{2}}u_2 + \dots, \quad \text{(A 3)}$$

$$w = \epsilon^{\frac{1}{2}}w_0 + \epsilon w_1 + \epsilon^{\frac{3}{2}}w_2 + \dots, \quad \text{(A 4)}$$

$$T = \epsilon^{\frac{1}{2}}T_0 + \epsilon T_1 + \epsilon^{\frac{3}{2}}T_2 + \dots, \quad \text{(A 5)}$$

as well as the expression for R in (3.1) and (3.2), and solve (A 1) by successive approximations for ψ. Then (3.4) determines u and w, and (2.1*d*) determines T. At order $\epsilon^{\frac{1}{2}}$ the solution is given by

$$\psi_0 = \frac{4i}{\pi}[A_0(X)\,e^{iq_0x} - A_0^*(X)\,e^{-iq_0x}]\sin\pi z, \quad \text{(A 6)}$$

$$u_0 = 4i[A_0(X)\,e^{iq_0x} - A_0^*(X)\,e^{-iq_0x}]\cos\pi z, \quad \text{(A 7)}$$

$$w_0 = 2\sqrt{2}\,[A_0(X)\,e^{iq_0x} + A_0^*(X)\,e^{-iq_0x}]\sin\pi z, \quad \text{(A 8)}$$

$$T_0 = 9\sqrt{2}\,\pi^2[A_0(X)\,e^{iq_0x} + A_0^*(X)\,e^{-iq_0x}]\sin\pi z, \quad \text{(A 9)}$$

where

$$q_0 = \pi/\sqrt{2}, \quad \text{(A 10)}$$

and $X = x\epsilon^{\frac{1}{2}}$ as in (3.3). The function $A_0(X)$ is arbitrary for the moment.

In expanding both sides of (A 1) it is necessary to consider the derivative $\partial/\partial x$ acting on x, u, w or T to be

$$\frac{\partial}{\partial x} \to \frac{\partial}{\partial x} + \epsilon^{\frac{1}{2}}\frac{\partial}{\partial X}, \quad \text{(A 11)}$$

where the $\partial/\partial x$ term acts on $\exp(\pm iq_0x)$, and $\partial/\partial X$ acting on functions of X is considered to be of order unity in the expansion. At order ϵ, (A 1) is solved by

$$\psi_1 = \frac{4i}{\pi}[A_1(X)\, e^{iq_0 x} - \text{c.c.}] \sin \pi z, \tag{A 12}$$

$$u_1 = 4i[A_1(X)\, e^{iq_0 x} - \text{c.c.}] \cos \pi z, \tag{A 13}$$

$$w_1 = 2\sqrt{2}\left[\left(A_1 - \frac{\sqrt{2}\, i}{\pi} A_0'\right) e^{iq_0 x} + \text{c.c.}\right] \sin \pi z, \tag{A 14}$$

$$T_1 = 9\sqrt{2}\, \pi^2 \left[\left(A_1 - \frac{\sqrt{2}\, i}{3\pi} A_0'\right) e^{iq_0 x} + \text{c.c.}\right] \sin \pi z - 18\pi |A_0|^2 \sin 2\pi z, \tag{A 15}$$

where c.c. denotes the complex conjugate, the prime denotes $\partial/\partial X$, and the function $A_1(X)$ is also arbitrary. At order $\epsilon^{\frac{3}{2}}$ the solution is

$$\psi_2 = \frac{4i}{\pi}[A_2(X)\, e^{iq_0 x} - \text{c.c.}] \sin \pi z - \frac{1}{\pi^3}(\tfrac{9}{8} + \sigma^{-1})[A_0' A_0^* + \text{c.c.}] \sin 2\pi z + \frac{18i}{427\pi^3}[A_0 |A_0|^2 e^{iq_0 x} - \text{c.c.}] \sin 3\pi z, \tag{A 16}$$

$$u_2 = 4i[A_2\, e^{iq_0 x} - \text{c.c.}] \cos \pi z - \frac{2}{\pi^2}(\tfrac{9}{8} + \sigma^{-1})[A_0' A_0^* + \text{c.c.}] \cos 2\pi z + \frac{54i}{427\pi^2}[A_0 |A_0|^2 e^{iq_0 x} - \text{c.c.}] \cos 3\pi z, \tag{A 17}$$

$$w_2 = 2\sqrt{2}\left[\left(A_2 - \frac{\sqrt{2}\, i}{\pi} A_1'\right) e^{iq_0 x} + \text{c.c.}\right] \sin \pi z + \frac{9\sqrt{2}}{427\pi^2}[A_0 |A_0|^2 e^{iq_0 x} + \text{c.c.}] \sin 3\pi z, \tag{A 18}$$

$$T_2 = [(9\sqrt{2}\, \pi^2 A_2 - 6i\pi A_1' + 24\sqrt{2}\, A_0 + 10\sqrt{2}\, A_0'' - 24\sqrt{2}\, A_0 |A_0|^2) e^{iq_0 x} + \text{c.c.}] \sin \pi z + [(12\sqrt{2}\, i A_0^* A_0' - 18\pi A_0 A_1^*) + \text{c.c.}] \sin 2\pi z + \frac{3249}{427\sqrt{2}}[A_0 |A_0|^2 e^{iq_0 x} + \text{c.c.}] \sin 3\pi z. \tag{A 19}$$

In addition, the solubility condition (Newell & Whitehead 1969) requires the function A_0 of (A 6–A 9) to satisfy the relation

$$A_0'' + A_0 - A_0 |A_0|^2 = 0. \tag{A 20}$$

At order ϵ^2 the terms proportional to $\exp(iq_0 x) \sin \pi z$ lead via the solubility condition to (4.5) and (4.6), with

$$k_1 = 2\sqrt{2}/\pi, \tag{A 21a}$$

$$k_2 = 22/9\sqrt{2}\, \pi, \tag{A 21b}$$

$$k_3 = (\tfrac{9}{8} + \sigma^{-1})(15\sigma^{-1} - 9)/36\sqrt{2}\, \pi, \tag{A 21c}$$

$$k_4 = 5k_5 = 25/3\sqrt{2}\, \pi, \tag{A 21d}$$

and to expressions for ψ_3, u_3, w_3 and T_3. In order to determine the phase of the core solution at order ϵ we shall require the equation satisfied by A_2, neglecting the derivative terms. To derive that equation we need the terms proportional to 1 and to $\exp(2iq_0 x)$ in ψ_3, u_3, w_3 and T_3 which do not involve derivatives. These are

$$\psi_3 = i e^{2iq_0 x} A_0^2 |A_0|^2 f_3(z) + \text{c.c.} + \ldots, \tag{A 22}$$

where f_3 is real and ... denotes terms involving derivatives with respect to X, as well as terms containing other functions of x. The functions u_3, w_3 and T_3 are

$$u_3 = i\, e^{2iq_0x} A_0^2 |A_0|^2 f_3'(z) + \text{c.c.} + \ldots, \tag{A 23}$$

$$w_3 = \sqrt{2}\, \pi\, e^{2iq_0x} A_0^2 |A_0|^2 f_3(z) + \text{c.c.} + \ldots, \tag{A 24}$$

$$T_3 = -18\pi(|A_1|^2 + A_0^* A_2 + A_0 A_2^*) \sin 2\pi z + |A_0|^4 f_4(z)$$
$$+ [A_0^2 |A_0|^2 e^{2iq_0x} f_5(z) + \text{c.c.}] + |A_0|^2 f_6(z) + \ldots, \tag{A 25}$$

where f_4 and f_5 are real. The equation for A_2 may now be found from the terms in (A 1) at order $\epsilon^{\frac{5}{2}}$, which are proportional to $\exp(iq_0 x) \sin \pi z$. One finds in the usual way

$$A_2'' + A_2 - 2A_0|A_1|^2 - A_0^* A_1^2 - 2A_2|A_0|^2 - A_0^2 A_0^* + k_6 |A_0|^4 A_0 + k_7 |A_0|^2 A_0 + \ldots = 0, \tag{A 26}$$

where ... denotes terms containing d/dX, and k_6 and k_7 are real constants.

It is instructive to recast the expansions (4.4)–(4.6) and (A 26) into a more general framework, starting from (A 1) and expanding in terms of the *independent* small parameters $|\psi|$, $\epsilon^{\frac{1}{2}}$, and d/dx. As mentioned in §1, the foregoing derivation can be repeated, and the result may be expressed in terms of the function

$$\Phi(x) \equiv \epsilon^{\frac{1}{2}} A_0(X) + \epsilon A_1(X) + \epsilon^{\frac{3}{2}} A_2(X) + \ldots \tag{A 27}$$

as

$$0 = \frac{d^2\Phi}{dx^2} + \epsilon\Phi - |\Phi|^2\Phi - ik_1\epsilon\frac{d\Phi}{dx} - ik_2\frac{d^3\Phi}{dx^3} + i(k_3+k_4)|\Phi|^2\frac{d\Phi}{dx}$$
$$+ i(k_3+k_5)\Phi^2\frac{d\Phi^*}{dx^3} + k_6\Phi|\Phi|^4 + k_7\epsilon\Phi|\Phi|^2 + \ldots, \tag{A 28}$$

where ... represents higher-order terms which we shall not need. The boundary condition is

$$0 = \Phi - \alpha_\pm \frac{d\Phi}{dx} - \beta_\pm \frac{d\Phi^*}{dx} + \ldots \quad (x = \pm L). \tag{A 29}$$

In the boundary layer we have $|\Phi|^2 = O(\epsilon)$, $d/dx = O(\epsilon^{\frac{1}{2}})$, and (A 28) and (A 29) agrees with (4.4)–(4.7). In the core region, we have $d/dx = O(\epsilon)$, and $|\Phi|^2 = O(\epsilon)$, from which the expansion of appendix E follows.

The present work is consistent with the analysis of Daniels (1978), where the formal expansion parameter was L^{-1}. In §3(iii) of that paper, expansions were made in L^{-1} with $\epsilon L^2 = O(1)$, i.e. $L^{-1} = O(\epsilon^{\frac{1}{2}})$. Apart from normalization, (4.4)–(4.7) above are equivalent to equations (3.40)–(3.44) of Daniels (1978), except that the value of α_{i} used in equation (3.44) of that paper is stated incorrectly. (The parameter α_{i} plays no significant role when $\delta = O(1)$ and is not used in the subsequent analysis in that paper.)

Appendix B. Sidewall boundary conditions

In this appendix we investigate the boundary condition on the envelope functions $A_0(X)$ and $A_1(X)$ at a sidewall, with dimensionless thickness t_{w} and thermal conductivity K_{w}, which is thermally clamped to the upper and lower plates at $z = 1, 0$, but with no heat loss to the outside. If we define a transverse coordinate $-t_{\text{w}} < x_{\text{w}} < 0$ in the wall, the temperature perturbation T_{w} from the basic conducting profile $-Rz$ satisfies

$$\nabla^2 T_{\text{w}} = 0. \tag{B 1}$$

The boundary conditions are

$$T_w = 0 \quad (z = 0, 1), \tag{B 2a}$$

$$\frac{dT_w}{dx_w} = 0 \quad (x_w = -t_w), \tag{B 2b}$$

together with equations relating T_w to the liquid temperature T at the point $x_w = 0$, $x = -L$,

$$T_w = T, \tag{B 3a}$$

$$\left.\frac{dT_w}{dx_w}\right|_{x=L} = \frac{K}{K_w}\left.\frac{dT}{dx}\right|_{x=L} \tag{B 3b}$$

Writing

$$T_w(x, z) = \sum_n T_w^{(n)}(x) \sin (n\pi z) \tag{B 4}$$

(and a similar expression in the liquid), the solutions to (B 1) and (B 2) are

$$T_w^{(n)}(x) \propto \cosh [n\pi(x_w + t_w)]. \tag{B 5}$$

Substituting into (B 3) leads to a boundary condition for each Fourier component of the liquid temperature:

$$T^{(n)} = \frac{\coth (n\pi t_w)}{n\pi} \frac{K}{K_w} \frac{dT^{(n)}}{dx} \quad (x = -L). \tag{B 6}$$

We shall only be interested in the boundary condition on the $n = 1$ component, which matches onto the amplitude functions A_0 and A_1,

$$\frac{dT^{(1)}}{dx} = \mu T^{(1)} \quad (x = -L), \tag{B 7}$$

where

$$\mu = \frac{K_w}{K} \pi \tanh (\pi t_w). \tag{B 8}$$

In the limit of a thin wall $t_w \ll d$, $\mu = \pi^2 K_w t_w / K$. For a thick wall the effective t_w in this expression becomes π^{-1}.

To derive boundary conditions on the amplitudes A_0 and A_1 we solve for the hydrodynamic variables near the sidewall (explicitly near $x = -L$) onto which these functions are to be matched. We follow the procedure of Daniels (1978), but include the effects of a finite wall conductivity via (B 7). Near the sidewall the velocities u and w are small, so that the hydrodynamic equations may be linearized. The solutions are

$$\psi(x) = \frac{4}{\pi} i\epsilon\{(B + \tilde{x}C)\, e^{iq_0 x} - \text{c.c.} + iDe^{-2\pi\tilde{x}}\} \sin \pi z + \dots, \tag{B 9a}$$

$$T(x) = 3\sqrt{2}\, \pi\epsilon\{[3\pi B - \sqrt{2}\, iC + 3\pi C\tilde{x}]\, e^{iq_0 x} + \text{c.c.} + 3\sqrt{2}\, \pi D e^{-2\pi\tilde{x}}\} \sin \pi z + \dots, \tag{B 9b}$$

where $\tilde{x} = x + L$, and there are other terms of higher order in $\epsilon^{\frac{1}{2}}$ or involving higher Fourier components that are not needed. The normalization in (B 9) is chosen for convenience in matching to the amplitude expansion. The complex constants B and C may now be related using the boundary conditions on the hydrodynamic variables: $u = w = 0$ and (B 7) for T. These are three linear equations in the four unknowns (B, B^*, C, C^*) from which an explicit relationship for $B(C, C^*, \mu)$ may be calculated.

$$B = \frac{1}{18\pi} [-(2 - 18\bar{\mu}) - 5i\sqrt{2}]\, C + \frac{1}{18\pi} [+(2 + 14\bar{\mu}) + i\sqrt{2}\, (5 + 8\bar{\mu})]\, e^{i\sqrt{2}\pi L} C^*, \tag{B 10}$$

where

$$\bar{\mu} = (1 + 2\mu/\pi)^{-1}. \tag{B 11}$$

Note that $\bar{\mu} = 1$ corresponds to perfectly insulating and $\bar{\mu} = 0$ to perfectly conducting boundaries.

The boundary conditions for A_0 and A_1 are given by matching the expressions (A 6)–(A 9) and (A 12)–(A 15) for $X \rightarrow \delta$ onto (B 9) for $\tilde{x} \gg 1$, where the exponentially decaying term is negligible. This leads to the correspondences

$$A_0(-\delta) = 0, \quad A_1(-\delta) \equiv B, \quad A_0'(-\delta) \equiv C. \tag{B 12}$$

A similar procedure may be repeated at $x = L$ ($X = \delta$). This gives the boundary conditions

$$A_0 = 0, \tag{B 13a}$$

$$A_1 - \alpha_\pm A_0' - \beta_\pm A_0^{*\prime} = 0 \quad (X = \pm\delta), \tag{B 13b}$$

with

$$\alpha_+ = -\alpha_-^* = \alpha, \quad \beta_+ = -\beta_-^* = \beta, \tag{B 14}$$

$$\alpha = \frac{1}{18\pi}[(2-18\bar{\mu}) + 5i\surd 2], \tag{B 15a}$$

$$\beta = \frac{1}{18\pi}[-(2+14\bar{\mu}) - i\surd 2\,(5+8\bar{\mu})]\,e^{-i\surd 2\pi L}. \tag{B 15b}$$

The relationships (B 14) may be understood by noting that the hydrodynamic equations and boundary conditions are symmetric under the transformations $x \rightarrow -x$, $u \rightarrow -u$, $w \rightarrow w$, $T \rightarrow T$.

Appendix C. Calculation of wavevectors in model II

The set of equations to be solved are

$$\rho_+ + \surd\tfrac{1}{2}\,\alpha\, e^{-i\gamma_+} + \surd\tfrac{1}{2}\,\beta\, e^{i(\gamma_+ - 2\theta_+)} = 0, \tag{C 1}$$

$$\rho_- + \surd\tfrac{1}{2}\,\alpha^* e^{i\gamma_-} + \surd\tfrac{1}{2}\,\beta^* e^{-i(\gamma_- + 2\theta_-)} = 0, \tag{C 2}$$

$$(\theta_+ - \theta_-) + 2n\pi = (\gamma_+ + \gamma_-) + 2\delta Q, \tag{C 3}$$

with

$$Q = \surd\tfrac{1}{2}\,\lambda\rho_+ \sin\gamma_+ = \surd\tfrac{1}{2}\,\lambda\rho_- \sin\gamma_-. \tag{C 4}$$

To solve, multiply (C 1) by $e^{i\gamma_+}$ and (C 2) by $e^{-i\gamma_-}$ and take sums and differences of real and imaginary parts. The imaginary parts lead to

$$\sin[\theta_+ - \gamma_+ + \theta_- + \gamma_-]\cos[\theta_+ - \gamma_+ - \theta_- - \gamma_- - \phi_\beta] = 0, \quad |\beta| \neq 0, \tag{C 5}$$

$$2Q + \lambda\alpha_{\mathrm{i}} - \lambda|\beta|\cos[\theta_+ - \gamma_+ + \theta_- - \gamma_-]\sin[\theta_+ - \gamma_+ - \theta_- - \gamma_- - \phi_\beta] = 0. \tag{C 6}$$

The first of these equations gives two classes of solutions.

$$\text{(i)}\quad \theta_+ - \gamma_+ + \theta_- + \gamma_- = n\pi, \tag{C 7}$$

so that (C 3) and (C 6) lead to the equation for the wavevector Q:

$$2Q + \lambda\alpha_{\mathrm{i}} \mp \lambda|\beta|\sin[2\delta Q - \phi_\beta] = 0, \tag{C 8}$$

where the minus sign refers to n even and the plus to n odd. The real parts of (C 1) and (C 2) can be used to calculate $\rho_+ = \rho_-$ and $\theta_+ = \theta_-$.

$$\text{(ii)}\quad \theta_+ - \theta_- - \gamma_+ - \gamma_- = \phi_\beta + (n+\tfrac{1}{2})\pi, \tag{C 9}$$

where n denotes any integer, not necessarily the same as in (C 3). Together with (C 3) this immediately gives for the wavevector

$$2\delta Q = \phi_\beta + (n+\tfrac{1}{2})\pi. \tag{C 10}$$

Then (C 6) becomes

$$2Q+\lambda\alpha_i-\lambda|\beta|(-1)^n\cos[\theta_+-\gamma_++\theta_-+\gamma_-]=0, \tag{C 11}$$

so that the range of Q for these solutions is limited by

$$|2Q+\lambda\alpha_i|\leqslant\lambda|\beta|.$$

It is readily checked that the real parts corresponding to (C 5) and (C 6) have solutions, but these are not as simple as in case (i), (C 8).

Appendix D. Solution of A_1 equation in the boundary layer

Let us insert (4.8) into (4.5) and (4.6), and define

$$A_1(X)=e^{i\phi_-}[C(\tilde{X})+iD(\tilde{X})], \tag{D 1}$$

with C and D real functions and $\tilde{X}=X+\delta$ as in (3.27). The function D satisfies the linear inhomogeneous equation

$$D''+D[1-\tanh^2(\surd\tfrac{1}{2}\tilde{X})]=K_1\operatorname{sech}^2(\surd\tfrac{1}{2}\tilde{X})+K_2\operatorname{sech}^4(\surd\tfrac{1}{2}\tilde{X}), \tag{D 2}$$

$$K_1=\surd\tfrac{1}{2}k_1+\surd2\,k_2-\surd\tfrac{1}{2}(2k_3+k_4+k_5), \tag{D 3}$$

$$K_2=-3\surd\tfrac{1}{2}k_2+\surd\tfrac{1}{2}(2k_3+k_4+k_5). \tag{D 4}$$

The complementary solutions of (D 2) are

$$D_1=\tanh(\surd\tfrac{1}{2}\tilde{X}), \tag{D 5}$$

$$D_2=\surd\tfrac{1}{2}\tilde{X}\tanh(\surd\tfrac{1}{2}\tilde{X})-1. \tag{D 6}$$

To find the particular solution we write

$$D=E_1(\tilde{X})\,D_1(\tilde{X})+E_2(\tilde{X})\,D_2(\tilde{X}), \tag{D 7}$$

and solve for E_1 and E_2. The result is

$$E_1=K_1\{\surd\tfrac{1}{2}\tilde{X}\operatorname{sech}^2(\surd\tfrac{1}{2}\tilde{X})+\tanh(\surd\tfrac{1}{2}\tilde{X})\}+K_2\{\tfrac{1}{2}\surd\tfrac{1}{2}\tilde{X}\operatorname{sech}^4(\surd\tfrac{1}{2}\tilde{X})$$
$$+\tfrac{1}{3}\operatorname{sech}^2(\surd\tfrac{1}{2}\tilde{X})\tanh(\surd\tfrac{1}{2}\tilde{X})+\tanh(\surd\tfrac{1}{2}\tilde{X})\}+c_-, \tag{D 8}$$

$$E_2=-K_1\operatorname{sech}^2(\surd\tfrac{1}{2}\tilde{X})-\tfrac{1}{2}K_2\operatorname{sech}^4(\surd\tfrac{1}{2}\tilde{X})+d_-, \tag{D 9}$$

with the constants c_- and d_- arbitrary. The function $C(\tilde{X})$ in (D 1) satisfies the equation

$$C''+C[1-3\tanh^2(\surd\tfrac{1}{2}\tilde{X})]=0, \tag{D 10}$$

with one solution equal to

$$C_1(\tilde{X})=\operatorname{sech}^2(\surd\tfrac{1}{2}\tilde{X}), \tag{D 11}$$

and the other exponentially large as $\tilde{X}\rightarrow\infty$. Thus only $C_1(\tilde{X})$ is retained. The full solution of (4.5) and (4.6) is therefore (4.9) with

$$\surd\tfrac{1}{2}B(\tilde{X})=D_1(\tilde{X})[E_1(\tilde{X})-c_-]+D_2(\tilde{X})[E_2(\tilde{X})-d_-]. \tag{D 12}$$

From (D 5), (D 6), (D 8) and (D 9) we find that $B(\tilde{X})<\infty$ as $\tilde{X}\rightarrow\infty$, as in (4.10*b*), and

$$B(\tilde{X}=0)\equiv b=\surd2\,(K_1+\tfrac{1}{2}K_2), \tag{D 13}$$

$$=\frac{\surd2}{576\pi}(145-63\sigma^{-1}-120\sigma^{-2}). \tag{D 14}$$

Appendix E. Core expansion

The core expansion may be generated from the hydrodynamic equations by repeating the analysis of appendix A with the scaling appropriate to the variable $\bar{X} = \epsilon x$, i.e. replacing (A 11) by the ansatz $\partial/\partial x \to \partial/\partial x + \epsilon\,\partial/\partial\bar{X}$, with $\partial/\partial\bar{X} = O(1)$. Alternatively, the core expansion may be generated from (A 28) by assuming $\partial/\partial x$ to be of order ϵ. In lowest order $(O(\epsilon^{\frac{3}{2}}))$ (A 28) yields

$$\bar{A}_0 = |\bar{A}_0|^2 \bar{A}_0, \tag{E 1}$$

whence

$$\bar{A}_0 = e^{i\bar{\theta}(\bar{X})}, \tag{E 2}$$

with $\bar{\theta}$ real. At $O(\epsilon^2)$ we find

$$\bar{A}_1 = 2\bar{A}_1|\bar{A}_0|^2 + \bar{A}_0^2\bar{A}_1^*. \tag{E 3}$$

Setting

$$\bar{A}_1 = \bar{r}_1(\bar{X})\,e^{i\bar{\theta}(\bar{X})}, \tag{E 4}$$

with $\bar{r}_1$ *not* assumed to be real, we find $\bar{r}_1 + \bar{r}_1^* = 0$, so

$$\bar{r}_1 = i\bar{r}_1, \quad \bar{r}_1 \text{ real}. \tag{E 5}$$

At $O(\epsilon^{\frac{5}{2}})$, (A 28) yields

$$0 = \bar{A}_2 + \bar{A}_0'' - 2\bar{A}_0|\bar{A}_1|^2 - \bar{A}_0^*\bar{A}_1^2 - 2\bar{A}_2|\bar{A}_0|^2 - \bar{A}_0^2\bar{A}_2^* + k_6\bar{A}_0|\bar{A}_0|^4 + k_7\bar{A}_0|\bar{A}_0|^2$$
$$- i\{k_1\bar{A}_0' - (k_3+k_4)|\bar{A}_0|^2\bar{A}_0' - (k_3+k_5)\bar{A}_0^2\bar{A}_0^{*\prime}\}. \tag{E 6}$$

Now let

$$\bar{A}_2 = \bar{r}_2\,e^{i\bar{\theta}(\bar{X})}, \tag{E 7}$$

with $\bar{\theta}$ given in (E 2). Then (E 6) implies

$$\bar{r}_2 + i\bar{\theta}'' - \bar{\theta}'^2 = 2\bar{r}_2 + \bar{r}_2^* - \bar{r}_1^2 + 2\bar{r}_1^2 + k_6 + k_7 - (k_1 - k_4 + k_5)\,\bar{\theta}', \tag{E 8}$$

from which it follows that $i\bar{\theta}''$ is real, i.e.

$$\bar{\theta}'' = 0, \tag{E 9}$$

$$\bar{\theta} = \bar{Q}\bar{X} + \bar{C}. \tag{E 10}$$

Equations (E 2) and (E 10) then yield (4.13).

Appendix F. The models of Pomeau & Manneville (1980)

F.1. *Derivation of the amplitude equations*

We may illustrate the general method for deriving amplitude equations from a microscopic model by considering (5.5*a*) or (5.5*b*) as the starting system. Let us set

$$u = \epsilon^{\frac{1}{2}}u_0 + \epsilon u_1 + \epsilon^{\frac{3}{2}}u_2 + \dots, \tag{F 1}$$

$$X = \epsilon^{\frac{1}{2}}x. \tag{F 2}$$

We wish to separate the functions $u_i(x)$ into a rapidly varying part with variation $\exp(inq_0x)$ and a slowly varying part depending on the variable X. We therefore divide the gradient ∂_x into two parts

$$\partial_x \to \partial_x + \epsilon^{\frac{1}{2}}\partial_X, \tag{F 3}$$

where the first term on the right-hand side of (F 3) acts only on the rapidly varying functions $\exp(inq_0x)$. The differential operator in (5.5) then takes the form

$$\partial_x^2 + q_0^2 \to \square + 2\epsilon^{\frac{1}{2}}\partial_x\partial_X + \epsilon\partial_X^2, \tag{F 4}$$

with
$$\Box = \partial_x^2 + q_0^2 \tag{F 5}$$
acting only on the rapidly varying functions. Inserting (F 3) and (F 4) into (5.5*a*) and expanding systematically in $\epsilon^{\frac{1}{2}}$, we find

$$O(\epsilon^{\frac{1}{2}}):\quad \Box^2 u_0 = 0, \tag{F 6}$$
$$O(\epsilon):\quad \Box^2 u_1 + 4\partial_x \partial_X \Box u_0 = 0, \tag{F 7}$$
$$O(\epsilon^{\frac{3}{2}}):\quad \Box^2 u_2 + 4\partial_x \partial_X \Box u_1 + (6\partial_x^2 \partial_X^2 + 2q_0^2 \partial_X^2 + u_0^2 - 1)\, u_0 = 0, \tag{F 8}$$
$$O(\epsilon^2):\quad \Box^2 u_3 + 4\partial_x \partial_X \Box u_2 + (6\partial_x^2 \partial_X^2 + 2q_0^2 \partial_X^2 + u_0^2 - 1)\, u_1 + (4\partial_x \partial_X^3 + 2u_0 u_1)\, u_0 = 0. \tag{F 9}$$

Equation (F 6) is solved by taking
$$u_0 = 3^{-\frac{1}{2}}[A_0(X)\, e^{iq_0 x} + \text{c.c.}], \tag{F 10}$$
where $A_0(X)$ is an arbitrary complex function and the factor $3^{-\frac{1}{2}}$ has been chosen for later convenience. Inserting (F 10) into (F 7) we see that (F 7) is solved by setting
$$u_1 = 3^{-\frac{1}{2}}[A_1(X)\, e^{iq_0 x} + \text{c.c.}]. \tag{F 11}$$
In order to solve (F 8) we write
$$u_2 = 3^{-\frac{1}{2}}[A_2(X)\, e^{iq_0 x} + \text{c.c.}] + 3^{-\frac{1}{2}}[B_2(X)\, e^{3iq_0 x} + \text{c.c.}], \tag{F 12}$$
and insert (F 10)–(F 12) into (F 8). Equating the terms proportional to $\exp(3iq_0 x)$ to zero, we find
$$B_2(X) = -\frac{A_0^3(X)}{192 q_0^4}. \tag{F 13}$$
The term proportional to $\exp(iq_0 x)$ has no contribution from the first two terms of (F 8), since
$$\Box e^{iq_0 x} = 0. \tag{F 14}$$
The third term in (F 8) yields the *amplitude equation*
$$4q_0^2 A_0'' + A_0 - |A_0|^2 A_0 = 0. \tag{F 15}$$
This is a condition on A_0 for the existence of a solution u_2 to (F 8), and is often called the 'solubility condition' (Newell & Whitehead 1969). Similarly, (F 9) is solved by setting
$$u_3 = 3^{-\frac{1}{2}}[A_3(X)\, e^{iq_0 x} + \text{c.c.}] + 3^{-\frac{1}{2}}[B_3(X)\, e^{3iq_0 x} + \text{c.c.}]. \tag{F 16}$$
The term proportional to $\exp(iq_0 x)$ in (F 9) now has contributions from the last two terms, which lead to the second amplitude equation
$$4q_0^2 A_1'' + A_1 - 2|A_0|^2 A_1 - A_0^2 A_1^* - 4iq_0 A_0''' = 0. \tag{F 17}$$
Introducing the coordinate
$$\bar{X} = (2q_0)^{-1} X, \tag{F 18}$$
these equations become
$$A_0'' + A_0 - |A_0|^2 A_0 = 0, \tag{F 19}$$
$$A_1'' + A_1 - 2|A_0|^2 A_1 - A_0^2 A_1^* - (i/2q_0^2)\, A_0''' = 0, \tag{F 20}$$
where the prime now denotes differentiation with respect to $\bar{X}$. These equations have the form (4.4)–(4.6) with
$$k_1 = k_3 = k_4 = k_5 = 0, \quad k_2 = (2q_0^2)^{-1}, \tag{F 21a}$$
corresponding to the parameter b of (D 13) equal to
$$b = (4q_0^2)^{-1}. \tag{F 21b}$$

Repeating the same procedure with (5.5*b*), we find

$$O(\epsilon^{\frac{1}{2}}): \quad \Box^2 u_0 = 0, \tag{F 22}$$

$$O(\epsilon): \quad \Box^2 u_1 + 4\Box\, \partial_X \partial_x u_0 + u_0 \partial_x u_0 = 0, \tag{F 23}$$

$$O(\epsilon^{\frac{3}{2}}): \quad \Box^2 u_2 + \partial_x u_1 + 4\Box\, \partial_X \partial_x u_1 + u_0 \partial_X u_0 + u_1 \partial_x u_1 - u_0 + 4\partial_X^2 \partial_x^2 u_0 + 2\Box\, \partial_X^2 u_0 = 0, \tag{F 24}$$

$$O(\epsilon^2): \quad \Box^2 u_3 + u_2 \partial_x u_0 + 4\Box\, \partial_X \partial_x u_2 + u_1 \partial_X u_0 + u_1 \partial_x u_1 - u_1 + 4\partial_X^2 \partial_x^2 u_1 + 2\Box\, \partial_X^2 u_1 + u_0 \partial_X u_1 + u_0 \partial_x u_2 + 4\partial_X^3 \partial_x u_0 = 0, \tag{F 25}$$

with solutions

$$u_0 = 3q_0[A_0 e^{iq_0 x} + \text{c.c.}], \tag{F 26}$$

$$u_1 = 3q_0[A_1 e^{iq_0 x} + \text{c.c.}] - \frac{i}{q_0}[A_0^2 e^{2iq_0 x} + \text{c.c.}], \tag{F 27}$$

$$u_2 = 3q_0[A_2 e^{iq_0 x} + \text{c.c.}] + \left[\left(\frac{13}{3q_0^2} A_0 A_0' - \frac{2i}{q_0} A_0 A_1\right) e^{2iq_0 x} + \text{c.c.}\right] - \left[\frac{9}{64q_0^3} A_0^3 e^{3iq_0 x} + \text{c.c.}\right] - \left[\frac{9}{q_0^2} A_0' A_0^* + \text{c.c.}\right]. \tag{F 28}$$

These functions, considered as functions of $\bar{X}$, (F 18), satisfy the amplitude equations

$$A_0'' + A_0 - |A_0|^2 A_0 = 0, \tag{F 29}$$

$$A_1'' + A_1 - 2|A_0|^2 A_1 - A_0^2 A_1^* - \frac{i}{2q_0^2} A'''_0 + \frac{10i}{3q_0^2}|A_0|^2 A_0' + \frac{5i}{q_0^2} A_0^2 A_0^{*\prime} = 0. \tag{F 30}$$

These are in the form (4.4)–(4.6) with

$$k_1 = 0, \quad k_2 = (2q_0^2)^{-1}, \quad k_3 + k_4 = \frac{10}{3q_0^2}, \quad k_3 + k_5 = \frac{5}{q_0^2}, \tag{F 31}$$

corresponding to the parameter b in (D 13) equal to

$$b = -47/12q_0^2. \tag{F 32}$$

F.2. *Derivation of boundary conditions*

Near the boundary the equations can be linearized and the solution written in the form

$$u = a_0 \epsilon[(B + \tilde{x}C) e^{iq_0 x} + \text{c.c.}], \tag{F 33}$$

where

$$\tilde{x} = x + L.$$

This solution, valid for $\tilde{x} = O(1)$, can be matched to the solution (F 1) which has the form

$$u = a_0 \epsilon^{\frac{1}{2}}[A_0(X) + \epsilon^{\frac{1}{2}} A_1(X)] e^{iq_0 x} + \text{c.c.} + \dots, \tag{F 34}$$

where $a_0 = 3^{-\frac{1}{2}}$ for model (*a*) and $a_0 = 3q_0$ for model (*b*). The boundary conditions (5.6) at $\tilde{x} = 0$ imply

$$B e^{-iq_0 L} + B^* e^{iq_0 L} = 0, \tag{F 35}$$

$$2iq_0 B e^{-iq_0 L} + C e^{-iq_0 L} + C^* e^{iq_0 L} = 0, \tag{F 36}$$

whence, by comparing (F 33) and (F 34), we find

$$A_1 - \alpha_\pm A_0' - \beta_\pm A_0^{*\prime} = 0, \quad X = \pm\delta = \pm L\epsilon^{\frac{1}{2}}, \tag{F 37}$$

with

$$\alpha_+ = -\alpha_-^* = \alpha = i/2q_0, \tag{F 38}$$

$$\beta_+ = -\beta_-^* = \beta = (i/2q_0)\, e^{2iq_0 L}. \tag{F 39}$$

In terms of the variable $\overline{X}$ ((F 18)) we have

$$\bar{\alpha} = i/4q_0^2, \tag{F 40}$$

$$\bar{\beta} = (i/4q_0^2)\, e^{2iq_0 L}. \tag{F 41}$$

F.3. *Phase-winding solutions*

The limiting wavenumbers (4.22) for phase-winding solutions coming from (F 19)–(F 21) and (F 29)–(F 32) are (dividing by $2q_0$ to return to the unbarred units)

$$\text{model } (a){:} \quad q_\pm = \tfrac{1}{2}\epsilon[-\alpha_i + b \pm |\beta|] = \pm\epsilon/16q_0^3, \tag{F 42}$$

$$\text{model } (b){:} \quad q_+ = -47\epsilon/48q_0^3, \tag{F 43}$$

$$q_- = -53\epsilon/48q_0^3. \tag{F 44}$$

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